Solutions of Practice Questions

Lecture#23

Q1.

Find all possible polar co-ordinates representations of the point P whose rectangular co-ordinates are $(3\sqrt{2}, -3\sqrt{2})$

Solution

We have

x= $3\sqrt{2}$ and $y = -3\sqrt{2}$

For polar coordinate $P(r,\theta)$

$$r = \sqrt{x^2 + y^2} = \sqrt{(3\sqrt{2})^2 + (-3\sqrt{2})^2} = \sqrt{18+18} = \sqrt{36} = \pm 6$$

$$\tan\theta = \frac{y}{x} = \frac{-3\sqrt{2}}{3\sqrt{2}} = -1 \Rightarrow \theta = \tan^{-1}(-1)$$

$$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$$

The point P $(3\sqrt{2}, -3\sqrt{2})$ lies in the fourth quadrant

When

$$\theta = \frac{3\pi}{4} \qquad r = -6$$

When $\theta = \frac{7\pi}{4}$ $\quad r = 6$

Polar co-ordinates of P are $(-6, \frac{3\pi}{4})$ and $(6, \frac{7\pi}{4})$

Q2.

Find polar co-ordinates of the point whose rectangular co-ordinates are $(2, -2)$

Solution

As we know

$r = \sqrt{x^2 + y^2}$ *and* $\quad \tan\theta = \frac{y}{x}$

So

$r = \sqrt{(2)^2 + (-2)^2} = \sqrt{8} = \pm 2\sqrt{2}$

$\tan\theta = \frac{-2}{2} = 1$

$\theta = \tan^{-1}(-1)$

$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

$\left(2\sqrt{2}\ ,\ \frac{7\pi}{4} + 2n\pi\right), \left(-2\sqrt{2}\ ,\ \frac{3\pi}{4} + 2n\pi\right)$

$n = 0, 1, 2, -------$

Q3.

Express the equation $x^2 = 8y$ in polar coordinates.

Solution

We use the formula $x=r\cos\theta$ and $y=r\sin\theta$

$$x^2 = 8y$$
$$(r\cos\theta)^2 = 8r\sin\theta$$
$$r^2\cos^2\theta = 8r\sin\theta$$
$$r\cos^2\theta = 8\sin\theta$$
$$r = \frac{8\sin\theta}{\cos^2\theta} = 8\sec\theta\tan\theta$$

Q4.

Convert the point (2, π/3) from polar to Cartesian coordinates.

Solution

$$(r,\theta) = (2,\frac{\pi}{3})$$

we have

$$x=r\cos\theta=2\cos\frac{\pi}{3}=2\frac{1}{2}=1$$

$$y=r\sin\theta=2\sin\frac{\pi}{3}=2\frac{\sqrt{3}}{2}=\sqrt{3}$$

Therefore, the point is $(1,\sqrt{3})$ in Cartesian co-ordinates.

Lecture#24

Q1.

Equation of a curve in polar co-ordinates is $r = 1 - \cos\theta$. Discuss the symmetry of graph of this curve about initial line.

Solution

Symmetry about initial line

As we know' $\cos(-\theta) = \cos\theta$ so equation of a curve remains same if θ is replaced by $-\theta$. So the curve is symmetrical about initial line.

Q2.

Equation of a curve in polar co-ordinates is $r^2 = a^2 \cos 2\theta$. Discuss the symmetry of the graph of this curve about y-axis.

Solution

Symmetry about y-axis

As we know

$$\begin{aligned}\cos\left(2(\pi-\theta)\right) &= \cos\left(2\pi - 2\theta\right) \\ &= \cos 2\pi \ \cos 2\theta + \sin 2\pi \ \ \sin 2\theta \\ &= \cos(2\theta)\end{aligned}$$

So equation of a curve remains same if θ is replaced by $\pi - \theta$. So the curve is symmetrical about y-axis.

Q3.

Equation of a curve in polar co-ordinates is $r^2 = 8\sin 2\theta$. Discuss the symmetry of the graph of this curve about y-axis.

Solution

Symmetry about y-axis

As we know

$$\begin{aligned}\sin\left(2(\pi-\theta)\right) &= \sin\left(2\pi - 2\theta\right)\\ &= \sin 2\pi\ \cos 2\theta - \cos 2\pi\ \ \sin 2\theta\\ &= -\sin(2\theta)\end{aligned}$$

So equation of a curve does not remain same if θ is replaced by $\pi - \theta$. So the curve is not symmetrical about y-axis.

Lecture#25

Q1.

Use double integrals in polar co-ordinates to find area of the region bounded by the curve $r = 7\cos\theta$.

Solution

Here $r = 7\cos\theta$ is a circle of radius $\frac{7}{2}$ and center at $\left(\frac{7}{2}, 0\right)$

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{7\cos\theta} r\ dr\ d\theta = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left|\frac{r^2}{2}\right|_0^{7\cos\theta} d\theta$$

$$= \frac{1}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (7\cos\theta)^2\ d\theta = \frac{49}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\ d\theta$$

$$= \frac{49}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\ d\theta$$

$$= \frac{49}{4}\int_{0}^{\pi} 1+\cos 2\theta\ d\theta$$

$$= \frac{49}{4}\left|\theta + \frac{\sin 2\theta}{2}\right|_{\frac{-\pi}{2}}^{\frac{\pi}{2}} =$$

$$= \frac{49}{4}\left(\frac{\pi}{2} - \frac{(-\pi)}{2}\right) = \frac{49}{4}(\frac{2\pi}{2}) = \frac{49\pi}{4}$$

Q2.

In each part, rewrite the double integral as an iterated integral in polar coordinates. (Do not evaluate.)

(a) $\int\int_R \sqrt{1-x^2-y^2}\,dxdy$, where R is the left half of the unit disk.

(b) $\int\int_R x^2 dxdy$ Where R is the right half of the ring $4 \le x^2+y^2 \le 9$

Solution

(a) The region R is the polar rectangle $\pi/2 \le \theta \le 3\pi/2$, $0 \le r \le 1$. In polar coordinates, the integrand is $\sqrt{1-r^2}$. So, we can rewrite the double integral as an iterated integral $\int_{\pi/2}^{3\pi/2}\int_0^1 \left(\sqrt{1-r^2}\right) r dr d\theta.$

(b) The region R is the polar rectangle $-\pi/2 \le \theta \le \pi/2$, $2 \le r \le 3$. In polar coordinates, the integrand is $(r\cos\theta)^2$. So, we can rewrite the double integral as an iterated integral

$$\int_{-\pi/2}^{\pi/2}\int_{2}^{3}(r\cos\theta)^2 r dr d\theta = \int_{-\pi/2}^{\pi/2}\int_{2}^{3} r^3\cos^2\theta dr d\theta .$$

Q3.Evaluate the following double integral by changing to polar coordinates.

$$\int_{0}^{1}\int_{0}^{\sqrt{2-x^2}}(x^2+y^2)\,\mathrm{dydx}.$$

Solution

$$\int_{0}^{1}\int_{0}^{\sqrt{2-x^2}}\left(x^2+y^2\right) dy\ dx$$

Region of integration is the part of a circle center at origin and radius $\sqrt{2}$ in first quadrant

$$\int_{0}^{\frac{\pi}{2}}\int_{0}^{\sqrt{2}} r^2\, r\ dr\ d\theta, \text{since } x^2+y^2=r^2$$

$$=\int_{0}^{\frac{\pi}{2}}\int_{0}^{\sqrt{2}} r^3\ dr\ d\theta$$

$$=\int_{0}^{\frac{\pi}{2}}\left|\frac{r^4}{4}\right|_0^{\sqrt{2}} d\theta = \frac{1}{4}\int_{0}^{\frac{\pi}{2}}\left|r^4\right|_0^{\sqrt{2}} d\theta = \frac{1}{4}\int_{0}^{\frac{\pi}{2}} 4\ d\theta$$

$$=\int_{0}^{\frac{\pi}{2}} d\theta = \left|\theta\right|_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

Q4. Evaluate $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\sin(x^2+y^2)\,\mathrm{dydx}$ by converting to polar coordinates.

Solution

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sin\left(x^2+y^2\right)\,dy\,dx$$

Region of integration is the part of a circle center at origin and radius 3 in upper half of plane

$$=\int_{0}^{\pi}\int_{0}^{3}(\sin r^2)r\;dr\;d\theta, \text{since } x^2+y^2=r^2$$

$$=\int_{0}^{\pi}-\frac{1}{2}\left|\cos r^2\right|_0^3\;d\theta=-\frac{1}{2}\int_{0}^{\pi}(\cos 9-1)\;d\theta$$

$$=-\frac{1}{2}\left|\theta\cos 9-\theta\right|_0^{\pi}=-\frac{\pi}{2}(\cos 9-1)=\frac{\pi}{2}(1-\cos 9)$$

Solution of Practice Questions

Lecture#26

Q1. Given the integral $\iiint_G (x^3+y+z^2)dzdydx$ where $-2\le z\le 2,\ 0\le y\le 3,\ 0\le x\le 2.$ Evaluate the integral with respect to the variable z only.

Solution

$$\int_0^2\int_0^3\int_{-2}^2 (x^3+y+z^2)dzdydx=\int_0^2\int_0^3\left[x^3z+yz+\frac{z^3}{3}\right]_{-2}^{2}dydx$$

$$=\int_0^2\int_0^3\left[4x^3+4y+\frac{16}{3}\right]dydx$$

Q2. Evaluate the double integral $\int_{-1}^{1}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^2+y^2+1)\;dydx$ by changing to polar co-ordinates.

Solution

$$\int_{-1}^{1}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^2+y^2+1)\,dy\,dx$$

$-1 \le x \le 1$, $-\sqrt{9-x^2} \le y \le \sqrt{9-x^2}$

$y = \sqrt{9-x^2} \Rightarrow x^2+y^2 = 9$

Circle center at origin and radius 3.So r varies from 0 to 3 and θ varies from 0 to 2π

$Also\ x^2+y^2+1 = r^2+1$

Putting values

$$\int_0^{2\pi}\int_0^3 (r^2+1)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^3 (r^3+r)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left|\frac{r^4}{4}+\frac{r^2}{2}\right|_0^3 d\theta$$

$$= \int_0^{2\pi}\left(\frac{81}{4}+\frac{9}{2}\right)d\theta = \frac{99}{4}(2\pi-0) = \frac{99\pi}{2}$$

Q3. Evaluate the double integral $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\,dydx$ by changing to polar co-ordinates.

Solution

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\,dydx$$

$-1 \le x \le 1$, $0 \le y \le \sqrt{1-x^2}$

$y = \sqrt{1-x^2} \Rightarrow x^2 + y^2 = 1$

Circle center at origin and radius 1. So r varies from 0 to 1 and θ varies from 0 to π

$Also \sqrt{x^2+y^2} = r$

Putting values

$$\int_0^{\pi}\int_0^{1} r\ r\ dr\ d\theta = \int_0^{\pi}\int_0^{1} r^2\ dr\ d\theta$$

$$= \int_0^{\pi} \left|\frac{r^3}{3}\right|_0^1 d\theta = \int_0^{\pi} \frac{1}{3} d\theta = \frac{1}{3}|\theta|_0^{\pi} = \frac{1}{3}(\pi) = \frac{\pi}{3}$$

Q4. Convert the following integral from Cartesian to polar coordinates. (Do not integrate it).

$$\int_0^{1}\int_0^{\sqrt{a^2-x^2}} x^2(x^2+y^2)^2\ dy\ dx.$$

Solution

$y = \sqrt{a^2 - x^2}$

$\Rightarrow y^2 = a^2 - x^2$ or $x^2 + y^2 = a^2$

Comparing it with the general equation of circle $x^2 + y^2 = r^2$

we get $r = a$, so we have $0 \le r \le a$ and θ varies from 0 to $\frac{\pi}{2}$,

$$\int_0^{\frac{\pi}{2}}\int_0^{a} r^2 \cos^2\theta (r^2)^2 r dr d\theta = \int_0^{\frac{\pi}{2}}\int_0^{a} r^7 \cos^2\theta dr d\theta$$

Lecture#27

Q1. If $r(t) = 5\hat{i} + \sin t\,\hat{j} + \cos t\,\hat{k}$ then find $\|r(t)\|$.

Solution

$$\begin{aligned}\|r(t)\| &= \sqrt{(5)^2 + (\sin t)^2 + (\cos t)^2} \\ &= \sqrt{25 + \cos^2 t + \sin^2 t} \\ &= \sqrt{25+1} = \sqrt{26}\end{aligned}$$

Q2. If the vector valued function $r(t) = e^{2t}\hat{i} + \ln(t-1)\,\hat{j} + t\hat{k}$, then find the vector that $r(t)$ associates with $t = 2$.

Solution

The vector associated with $t = 2$ is

$$\begin{aligned}r(2) &= e^{2(2)}\hat{i} + \ln(2-1)\,\hat{j} + 2\hat{k} \\ &= e^4\hat{i} + \ln(1)\,\hat{j} + 2\hat{k} \\ &= e^4\hat{i} + 2\hat{k}\end{aligned}$$

Lecture#28

Q1. Check whether the given limit exists or not? Also justify your answer.

$$\lim_{t\to 0}\left[\left(\frac{1}{t}\right)\hat{i}+e^{2t}\,\hat{j}+\cos t\,\hat{k}\right].$$

Solution

$$\lim_{t\to 0}\left[\left(\frac{1}{t}\right)\hat{i}+e^{2t}\,\hat{j}+\cos t\,\hat{k}\right]=\lim_{t\to 0}\left(\frac{1}{t}\right)\hat{i}+\lim_{t\to 0}e^{2t}\,\hat{j}+\lim_{t\to 0}\cos t\,\hat{k}$$

Since the limit of first component, that is, $\lim_{t\to 0}\left(\frac{1}{t}\right)$ does not exist. So even though the limit of remaining two components exist, the limit of vector-valued function does not exist.

Q2. The vector equation for a given curve C is $\vec{r}(t)=\left(\frac{1}{t}\right)\hat{i}+t^2\,\hat{j}+(2\sqrt{t})\,\hat{k}$ and its derivative is $\vec{r}'(t)=-\frac{1}{t^2}\hat{i}+2t\,\hat{j}+\frac{1}{\sqrt{t}}\hat{k}$. Find the tangent vector to C at the point $t_\circ=3$.

Solution

At $t_\circ=3$

$$\vec{r}(3)=\frac{1}{3}\hat{i}+9\,\hat{j}+2\sqrt{3}\,\hat{k} \quad \text{and} \quad \vec{r}'(3)=-\frac{1}{9}\hat{i}+6\,\hat{j}+\frac{1}{\sqrt{3}}\hat{k}$$

For vector equation of tangent line at the point where $t_\circ=3$, we have

$$\vec{r} = \vec{r}(3) + t\,\vec{r}'(3)$$
$$= \left(\frac{1}{3}\hat{i} + 9\hat{j} + 2\sqrt{3}\hat{k}\right) + t\left(-\frac{1}{9}\hat{i} + 6\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right)$$
$$= \left(\frac{1}{3} - \frac{1}{9}t\right)\hat{i} + \left(9 + 6t\right)\hat{j} + \left(2\sqrt{3} + \frac{1}{\sqrt{3}}t\right)\hat{k}$$

Q3.

Let $\vec{r(t)} = \sin t\,\hat{i} + 3\cos t\,\hat{j}$. Find *t*, such that $\vec{r(t)}$ and $\vec{r'(t)}$ are perpendicular to each other.

Solution

$\vec{r(t)} = \sin t\,\hat{i} + 3\cos t\,\hat{j}$ *and* $\vec{r'(t)} = \cos t\,\hat{i} - 3\sin t\,\hat{j}$

Since $\vec{r(t)}$ *and* $\vec{r'(t)}$ *are perpendicular so*

$$\vec{r(t)} \bullet \vec{r'(t)} = 0$$
$$\left(\sin t\,\hat{i} + 3\cos t\,\hat{j}\right) \bullet \left(\cos t\,\hat{i} - 3\sin t\,\hat{j}\right) = 0$$
$$\sin t\cos t - 9\sin t\cos t = 0$$
$$-8\sin t\cos t = 0$$
$$\sin t\cos t = 0$$

Multiply by 2 *on both sides*

$$2\sin t\cos t = 0$$
$$\sin 2t = 0$$
$$2t = 0,\ \pi,\ 2\pi,\ 3\pi,\ ---$$
$$t = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ ---$$
$$t = \frac{n\pi}{2}; n = 0,\ 1,\ 2,\ 3,\ ---$$

Lecture#29

Q#1: Write down the expression for the arc-length of the curve represented by the vector valued function

$\vec{r}(t) = (5t - \ln 3t)\hat{i} + (t^2 + 7)\hat{j}$ where $0 \le t \le 4$. (Do not evaluate the expression).

Solution:

Here

$$\mathrm{x} = 5t - \ln 3t, \quad y = t^2 + 7$$
$$\frac{dx}{dt} = 5 - \frac{3}{3t}, \quad \frac{dy}{dt} = 2t$$

$$\text{Arc length} = \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$
$$= \int_0^4 \sqrt{\left(5 - \frac{3}{3t}\right)^2 + (2t)^2}$$

Q#2: What is the arc-length of the curve $\vec{r}(t) = 2\sin t\,\hat{i} + 2\cos t\,\hat{j}$; when $0 \le t \le 2\pi$?

Solution:

$\vec{r}(t) = 2\ \sin t\ \hat{i} + 2\cos t\ \hat{j}$

Parametric equations corresponding to the vector equation are

$x = 2\sin t\ ,\ \ y = 2\cos t$

$\frac{dx}{dt} = 2\cos t, \quad \frac{dy}{dt} = -2\sin t$

$$Arc-length = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= \int_0^{2\pi} \sqrt{\left(2\cos t\right)^2 + \left(-2\sin t\right)^2}\ dt$$

$$= \int_0^{2\pi} \sqrt{\left(4\cos^2 t\right) + \left(4\sin^2 t\right)}\ dt$$

$$= \int_0^{2\pi} \sqrt{4\left(\sin^2 t + \cos^2 t\right)}\ dt$$

$$= 2\int_0^{2\pi} dt\ \ = 2\left|t\right|_0^{2\pi} = \ 4\ \pi$$

Q#3: What is the arc-length of the curve $\vec{r}(t) = (2+5t)\ \hat{i} + (3-6t)\ \hat{j}$ when $2 \le t \le 3$?

Solution:

$\vec{r}(t) = (2+5t)\,\hat{i} + (3-6t)\,\hat{j}$

Parametric equations corresponding to the vec tor equation are

$x = 2+5t, \quad y = 3-6t$

$\frac{dx}{dt} = 5, \quad \frac{dy}{dt} = -6$

$$Arc-length = \int_2^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_2^3 \sqrt{(5)^2 + (-6)^2}\, dt$$

$$= \int_2^3 \sqrt{(25) + (36)}\, dt$$

$$= \int_2^3 \sqrt{61}\, dt \quad = \sqrt{61}\int_2^3 dt$$

$$= \sqrt{61}\,|t|_2^3 = \sqrt{61}(3-2) = \sqrt{61}$$

Q#4: Given the equations of two curves $y = x$ and $y = \sqrt{x}$. Find the intersecting points of these curves.

Solution:

$x = \sqrt{x}$

$x^2 = x \quad or \quad x^2 - x = 0$

$x(x-1) = 0$

$x = 0$ or $x = 1$

$x = 0$ and $x = 1$ are the intersecting points.

Q#5: Given the equations of two curves $y = -6 - x^2$ and $y = 4 - 7x$. For which values of x these curves intersect each other?

Solution:

$-6-x^2=4-7x$

$x^2-7x+10=0$

$x^2-5x-2x+10=0$

$x(x-5)-2(x-5)=0$

$(x-2)(x-5)=0$

$x=2$ and $x=5$ are the required insecting points.

Q#6: Given the equations of two curves $y=4-x$ and $y=4-x^2$. For which values of x these curves intersect each other?

Solution:

$4-x=4-x^2$

$x^2-x=0$

$x(x-1)=0$

$x=0$ or $x-1=0$

$x=0$ and $x=1$ are the intersecting points.

Lecture#30

Q#1: Determine whether the following differential is exact or not.

$$dz=4xy^3\ dx+6x^2y^2\ dy$$

Solution:

$P = 4xy^3$

$Q = 6x^2y^2$

$\frac{\partial P}{\partial y} = 12xy^2$

$\frac{\partial Q}{\partial x} = 12xy^2$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

So the given differential is exact.

Q#2: Determine whether the following differential is exact or not.

$$dz = (5x + 9xy)\ dx + 5x^3\, dy$$

Solution:

Let $P = 5x + 9xy$ and $Q = 5x^3$

Then

$\frac{\partial P}{\partial y} = 9x$ $\qquad$ $\frac{\partial Q}{\partial x} = 15x^2$

Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ so given differential is not exact.

Q#3: Determine whether the following differential is exact? If so, find z.

$$dz = 3x(\mathrm{x}\, y - 2)\ dx + (x^3 + 2y)\ dy$$

Solution:

Since $P = 3x(\text{x}\, y - 2)$ and $Q = (x^3 + 2y)$

Then

$\frac{\partial P}{\partial y} = 3x^2$ $\qquad$ $\frac{\partial Q}{\partial x} = 3x^2$

Since the given differential is exact, therefore, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Since dz is exact differential so

$$z = \int P\ dx = \int 3x(\text{x}\, y - 2)\ dx = \int 3x^2 y - 6x)\ dx$$
$$= x^3 y - 3x + f(y) - - - - - - - (1)$$

$$z = \int Q\ dy = \int (x^3 + 2y)\, dy$$
$$= x^3 y + y^2 + f(x) - - - - - - - - - - - - (2)$$

Compare right hand side of both the expressions (1) and (2) we get

$f(y) = y^2$

and

$f(x) = -3x$

Thus

$z = x^3 y - 3x + y^2$

Lecture#31

Q#1: Evaluate the line integral $\int_C -y\ dx + x\ dy$ along $y^2 = 3x$ from the point $(3, 4)$ to the point $(1, 0)$.

Solution:

$$y^2 = 3x$$

$$x = \frac{y^2}{3}$$

$$dx = \frac{2}{3}\ y\ dy$$

Put values of C , x and dx in given line integral

$$\int_C -y\ dx + x\ dy$$

$$\int_4^0 -y\left(\frac{2}{3}y\,dy\right) + \frac{y^2}{3}dy = \int_4^0 -\frac{2}{3}\ y^2\ dy + \frac{1}{3}\ y^2\ dy$$

$$= \int_4^0 \frac{-1}{3}\ y^2\ dy$$

$$= \frac{-1}{3}\left|\frac{y^3}{3}\right|_4^0$$

$$= \frac{-1}{9}(0 - 64)$$

$$= \frac{64}{9}$$

Q#2: Evaluate the line integral $\int_C 3y^2\ dy$ where C is the line segment from $(2, 0)$ to $(1, 5)$.

Solution:

$$\int_C 3y^2\ dy = \int_0^5 3y^2\ dy$$

$$= \left|y^3\right|_0^5$$

$$= 125$$

Q#3: Evaluate the line integral $\int_C 3x^2 y\ dx + xy\ dy$ along $y = x^3 + 1$ from the point $(-1, 0)$ to the point $(1, 2)$.

Solution:

$y = x^3 + 1$

$dy = 3x^2\,dx$

Put values of C, y and dy in given line integral

$$\int_C 3x^2 y\,dx + xy\,dy$$

$$\int_{-1}^{1} 3x^2(x^3+1)\,dx + x(x^3+1)(3x^2)\,dx = \int_{-1}^{1} \left(3x^5 + 3x^2 + 3x^6 + 3x^3\right)dx$$

$$= \left| \frac{x^6}{2} + x^3 + \frac{3}{7}x^7 + \frac{3}{4}x^4 \right|_{-1}^{1}$$

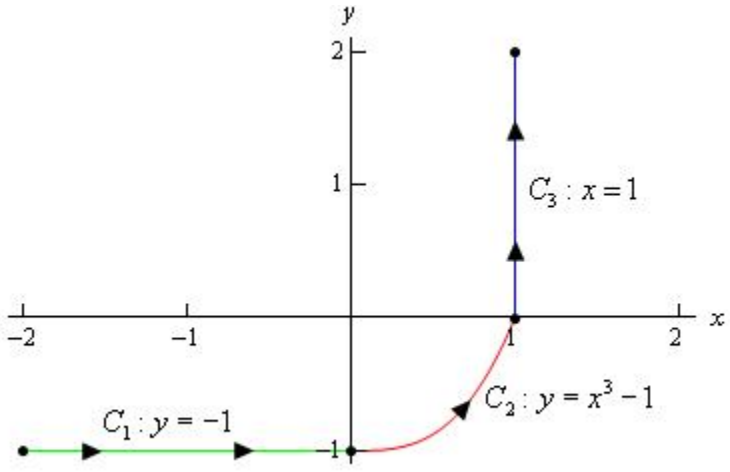
$$= \frac{1}{2} + 1 + \frac{3}{7} + \frac{3}{4} - \frac{1}{2} + 1 + \frac{3}{7} - \frac{3}{4}$$

$$= 2 + \frac{6}{7}$$

$$= \frac{14+6}{7} = \frac{20}{7}$$

Lecture#32

Q1. Evaluate $\int_C 4x^3 ds$ where C is the curve shown below.



Solution:

So, first we need to parameterize each of the curves

$$C_1: \; x = t, \; y = -1, \quad -2 \le t \le 0$$
$$C_2: \; x = t, \; y = t^3 - 1, \quad 0 \le t \le 1$$
$$C_3: \; x = 1, \; y = t, \quad 0 \le t \le 2$$

Now let's do the line integral over each of these curves.

$$\int_{C_1} 4x^3 \, ds = \int_{-2}^{0} 4t^3 \sqrt{(1)^2 + (0)^2} \, dt = \int_{-2}^{0} 4t^3 \, dt = t^4 \Big|_{-2}^{0} = -16$$

$$\begin{aligned} \int_{C_2} 4x^3 \, ds &= \int_0^1 4t^3 \sqrt{(1)^2 + (3t^2)^2} \, dt \\ &= \int_0^1 4t^3 \sqrt{1 + 9t^4} \, dt \\ &= \frac{1}{9}\left(\frac{2}{3}\right)\left(1 + 9t^4\right)^{\frac{3}{2}} \Big|_0^1 = \frac{2}{27}\left(10^{\frac{3}{2}} - 1\right) = 2.268 \end{aligned}$$

$$\int_{C_3} 4x^3 \, ds = \int_0^2 4(1)^3 \sqrt{(0)^2 + (1)^2} \, dt = \int_0^2 4 \, dt = 8$$

Finally, the line integral that we were asked to compute is,

$$\int_C 4x^3\,ds = \int_{C_1} 4x^3\,ds + \int_{C_2} 4x^3\,ds + \int_{C_3} 4x^3\,ds$$
$$= -16 + 2.268 + 8$$
$$= -5.732$$

Answer (-5.732)

Q2. Evaluate the following line integral with respect to arc-lengths

$\int_C 3xy^2z^2\,ds$, Where C is the curve $x = t \quad , \quad y = t^2 \quad , \quad z = \frac{2}{3}t^3 \quad and \quad 0 \le t \le 1$.

Solution:

$$x = t, \qquad y = t^2, \qquad z = \frac{2}{3}t^3$$
$$\frac{dx}{dt} = 1, \qquad \frac{dy}{dt} = 2t, \qquad \frac{dz}{dt} = 2t^2$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$
$$= \sqrt{1 + 4t^2 + 4t^4} = \sqrt{(1 + 2t^2)^2} = 1 + 2t^2$$

$$3xy^2z^2 = 3t(t^2)^2\left(\frac{2}{3}t^3\right)^2 = \frac{4}{3}t^{11} \quad (Putting\ values\ of\ x,\ y,\ z)$$

$$\int_C \left(3xy^2z^2\right)\, ds = \int_0^1 \left(\frac{4}{3}t^{11}\right)\left(1+2t^2\right)\, dt$$

$$= \int_0^1 \left(\frac{4}{3}t^{11} + \frac{8}{3}t^{13}\right)\, dt$$

$$= \frac{4}{3}\left|\frac{t^{12}}{12}\right|_0^1 + \frac{8}{3}\left|\frac{t^{14}}{14}\right|_0^1$$

$$= \frac{4}{3}\left(\frac{1}{12}\right) + \frac{8}{3}\left(\frac{1}{14}\right) = \frac{1}{9} + \frac{4}{21} = \frac{7+12}{63} = \frac{19}{63}$$

Answer $\left(\frac{19}{63}\right)$

Q3.

Evaluate the following line integral which is independent of path.

$$\int_{(2,-2)}^{(-1,0)} (3x^2y^2)\, dx + (2x^3y)\, dy$$

Solution:

Let $dz = (3x^2y^2)\, dx + (2x^3y)\, dy$

As given line integral is independent of path so dz is an exact differential. z will be found as given below

$$P(x,\ y) = 3x^2y^2, \qquad Q(x,\ y) = 2x^3y$$

$$z = \int P\ dx = \int \left(3x^2y^2\right) dx \qquad z = \int Q\ dy = \int \left(2x^3y\right) dy$$
$$= x^3y^2 + f(y) - - - -(1) \qquad = x^3y^2 + f(x) - - - -(2)$$

As there is no dissimilar term at the right side of equations (1) and (2), so

$f(y) = 0$ and $f(x) = 0$

Thus $z = x^3y^2$

So for $dz = (2xy^3)\ dx + (3y^2x^2)\ dy$, we have $z = x^3y^2$

Put this value in given integral, with limits $(2, -2)$ to $(-1, 0)$

$$\int_C (3x^2y^2)\ dx + (2x^3y)\ dy = \int_{(2,-2)}^{(-1,0)} d(x^3y^2)$$
$$= \left|x^3y^2\right|_{(2,-2)}^{(-1,0)}$$
$$= \left[8(4)\right] = 32$$

Answer (32)

Q4.

Evaluate the following line integral which is independent of path.

$$\int_{(0,0)}^{\left(1,\frac{\pi}{2}\right)} (2\,x \sin y)\ dx - (x^2 \cos y)\, dy$$

Solution:

Let $dz = (2\,x \sin y)\ dx - (x^2 \cos y)\ dy$

Since given line integral is independent of path so dz is an exact differential. z will be found as given below.

$$P(x,\ y) = 2\,\mathrm{x}\sin y, \quad Q(x,\ y) = -\mathrm{x}^2\cos y$$

$$z = \int P\ dx = \int 2\,\mathrm{x}\sin y\ dx$$
$$= x^2\sin y + f(y) - - - -(1)$$

$$z = \int Q\ dy = -\int \mathrm{x}^2\cos y dy$$
$$= \mathrm{x}^2\sin y + f(x) - - - -(2)$$

As there is no dissimilar term at the right side of equations (1) and (2) so

$$f(y) = 0 \ \text{ and } \ f(x) = 0$$
$$\text{Thus} \quad z = \mathrm{x}^2\sin y$$

So for $dz = (2\,\mathrm{x}\sin y)\ dx - (\mathrm{x}^2\cos y)\ dy$, we have $z = \mathrm{x}^2\sin y$

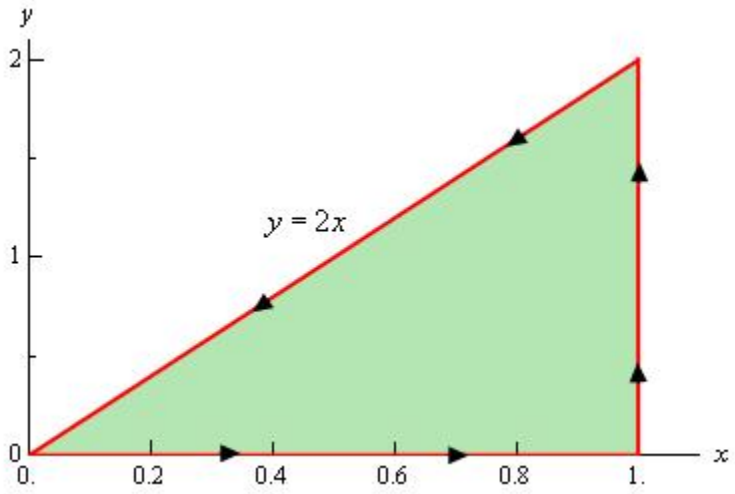
Put this value in given integral, with limits $(0,\ 0)$ to $\left(1,\ \frac{\pi}{2}\right)$

$$\int_C (2\,\mathrm{x}\sin y)\ dx - (\mathrm{x}^2\cos y)\ dy = \int_{(0,\ 0)}^{\left(1,\ \frac{\pi}{2}\right)} d(\mathrm{x}^2\sin y)$$
$$= \left|\mathrm{x}^2\sin y\right|_{(0,\ 0)}^{\left(1,\ \frac{\pi}{2}\right)}$$
$$= 1$$

Answer (1)

Lecture#33

Q1. Use Green's Theorem to evaluate $\oint_C xydx + x^2y^3dy$, where *C* is the triangle with vertices $(0,0),\ (1,0),\ (1,2)$ with positive orientation.



Solution:

So, the curve does satisfy the conditions of Green's Theorem and we can see that the following inequalities will define the region enclosed.

$0 \le x \le 1$ $\qquad$ $0 \le y \le 2x$

We can identify *P* and *Q* from the line integral. Here they are.

$$P = xy \qquad Q = x^2y^3$$

So, using Green's Theorem the line integral becomes,

$$\oint_C xy\,dx + x^2y^3\,dy = \iint_D 2xy^3 - x\,dA$$

$$= \int_0^1 \int_0^{2x} 2xy^3 - x\,dy\,dx$$

$$= \int_0^1 \left(\frac{1}{2}xy^4 - xy\right)\Big|_0^{2x} dx$$

$$= \int_0^1 8x^5 - 2x^2\,dx$$

$$= \left(\frac{4}{3}x^6 - \frac{2}{3}x^3\right)\Big|_0^1$$

$$= \frac{2}{3}$$

Answer ($\frac{2}{3}$)

Q2.

Use Green's Theorem to evaluate $\oint_C \left(x^2 - 4xy\right)dx + \left(y^2 - xy\right)dy$ where C is a square with vertices at $(0, 0)$, $(2, 0)$, $(2, 2)$ and $(0, 2)$

Solution:

Here

$P = \left(x^2 - 4xy\right)$

and

$Q = \left(y^2 - xy\right)$

$\frac{\partial P}{\partial y} = -4x$

$\frac{\partial Q}{\partial x} = -y$

Limits for x is from 0 to 2 and for y is $0 \ to \ 2.$.

Thus by Green's Theorem

$$\oint_C \left(x^2 - 4xy\right) dx + \left(y^2 - xy\right) dy = -\int_0^2 \int_0^2 (-4x \ - y) \ dy \ dx$$

$$= \int_0^2 \int_0^2 (4x \ + y) \ dy \ dx$$

$$= \int_0^2 \left| 4xy + \frac{y^2}{2} \right|_0^2 dx$$

$$= \int_0^2 \left(8x + 2\right) dx$$

$$= \left| 4x^2 + 2x \right|_0^2 = 16 + 4 = 20$$

Answer (20)

Q3. Use Green's Theorem to evaluate $\oint_C y^3 dx - x^3 dy$ where *C* is the positively oriented circle of radius 2 centered at the origin.

Solution:

Let's first identify *P* and *Q* from the line integral.

$P = y^3 \qquad Q = -x^3$

Now, using Green's theorem on the line integral gives,

$$\oint_C y^3 \, dx - x^3 \, dy = \iint_D -3x^2 - 3y^2 \, dA$$

Where *D* is a disk of radius 2 centered at the origin.

Since *D* is a disk it seems like the best way to do this integral is to use polar coordinates. Here is the evaluation of the integral.

$$\begin{aligned}\oint_C y^3\,dx - x^3\,dy &= -3\iint_D \left(x^2+y^2\right)dA \\ &= -3\int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta \\ &= -3\int_0^{2\pi}\frac{1}{4}r^4\Big|_0^2\,d\theta \\ &= -3\int_0^{2\pi} 4\,d\theta \\ &= -24\pi\end{aligned}$$

Answer (-24π)

Lecture#34

Q1. Find $curl\ \vec{F}$, if $\vec{F} = x^2y\,\hat{i} + xyz\,\hat{j} - x^2y^2\,\hat{k}$.

Solution:

$$\vec{F} = x^2 y\,\hat{i} + xyz\,\hat{j} - x^2 y^2\,\hat{k}$$

$$curl\ \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & xyz & -x^2 y^2 \end{vmatrix}$$

$$= \left(\frac{\partial(-x^2 y^2)}{\partial y} - \frac{\partial(xyz)}{\partial z} \right)\hat{i} + \left(\frac{\partial(x^2 y)}{\partial z} - \frac{\partial(-x^2 y^2)}{\partial x} \right)\hat{j} + \left(\frac{\partial(xyz)}{\partial x} - \frac{\partial(x^2 y)}{\partial y} \right)\hat{k}$$

$$= (-2x^2 y - xy)\,\hat{i} - 2xy^2\,\hat{j} + (yz - x^2)\,\hat{k}$$

Answer $((-2x^2 y - xy)\,\hat{i} - 2xy^2\,\hat{j} + (yz - x^2)k\,)$

Q2. Find $div\ \vec{F}$, if $\vec{F} = yz^2\,\hat{i} + xy\,\hat{j} + yz\,\hat{k}$

Solution:

$$\vec{F} = yz^2\,\hat{i} + xy\,\hat{j} + yz\,\hat{k}$$

$$div\,\vec{F} = \nabla \bullet \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \bullet \left(\vec{F} = yz^2\,\hat{i} + xy\,\hat{j} + yz\,\hat{k} \right)$$

$$= \frac{\partial(yz^2)}{\partial x} + \frac{\partial(xy)}{\partial y} + \frac{\partial(yz)}{\partial z}$$

$$= 0 + x + y = x + y$$

Answer $(x + y\ )$

Q3. If $A = x^3 y\hat{i} + y2z^3\hat{j} - z^3x^2\hat{k}$ then find grad div A .

Solution:

$$A = x^3 y\hat{i} + y^2z^3\hat{j} - z^3x^2\hat{k}$$

$$div\, A = \nabla.A = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(x^3 y\hat{i} + y^2z^3\hat{j} - z^3x^2\hat{k}\right)$$

$$= 3x^2y + 2yz^3 - 3z^2x^2 = \phi \text{ (say)}$$

$$\text{grad div } A = \text{grad}\,\phi$$

$$= \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)$$

$$= (6xy - 6xz^2)\hat{i} + (3x^2 + 2z^3)\hat{j} - (6yz^2 - 6zx^2)\hat{k}$$

$$= 6x(y - z^2)\hat{i} + (3x^2 + 2z^3)\hat{j} - 6z(yz - x^2)\hat{k}$$

Answer $(6x(y - z^2)\hat{i} + (3x^2 + 2z^3)\hat{j} - 6z(yz - x^2)\hat{k}$)

Q4: Given , f(x,y)=sinxy+cosxy find $div(\nabla F)$.

Solution:

We are given that

$$f(x, y) = \text{sinxy+cosxy}$$

$$\nabla f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right)(\text{sinxy+cosxy})$$

$$= (\text{ycosxy} - y\sin xy)\hat{i} + (\text{xcosxy} - x\sin xy)\hat{j}$$

$$grad(\nabla f) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right) \bullet \left((\text{ycosxy} - y\sin xy)\hat{i} + (\text{xcosxy} - x\sin xy)\hat{j}\right)$$

$$= -y^2 \sin xy - y^2 \cos xy - x^2 \sin xy - x^2 \cos xy$$

$$= -\sin xy(x^2 + y^2) - \cos xy(x^2 + y^2)$$

Answer $(-\sin xy(x^2 + y^2) - \cos xy(x^2 + y^2))$

Question 1

Evaluate $\int x^3 \ln x dx$ by parts.

Ans

$$\frac{x^4}{4}\ln x - \frac{x^4}{16} + c$$

Question 2

Evaluate $\int x\cos x dx$ by parts.

Ans

$$x\sin x + \cos x + c$$

Question 3

Evaluate $\int_0^{\frac{\pi}{2}} \sin^{11} x dx$

$$\frac{10}{11}.\frac{8}{9}.\frac{6}{7}.\frac{4}{5}.\frac{2}{3}$$

Lecture#35

1. Use Wallis cosine formula to evaluate $\int_0^{\frac{\pi}{2}} \left(\cos^3 x + \cos^5 x\right) dx$.

   Solution

   $$\int_0^{\frac{\pi}{2}} \left(\cos^3 x + \cos^5 x\right) dx = \int_0^{\frac{\pi}{2}} \left(\cos^3 x\right) dx + \int_0^{\frac{\pi}{2}} \left(\cos^5 x\right) dx$$

   $$\int_0^{\frac{\pi}{2}} \cos^n x \, dx = \frac{2.4.6.....(n-1)}{3.5.7.....n} \text{ if n is odd and } \geq 3$$

   $$\int_0^{\frac{\pi}{2}} \cos^3 x \, dx = \frac{2}{3}$$

   $$\int_0^{\frac{\pi}{2}} \cos^5 x \, dx = \frac{2\times 4}{3\times 5} = \frac{8}{15}$$

   $$\int_0^{\frac{\pi}{2}} \left(\cos^3 x + \cos^5 x\right) dx = \frac{2}{3} + \frac{8}{15} = \frac{18}{15}$$

2. Use Wallis sine formula to evaluate $\int_0^{\frac{\pi}{2}} \left(\sin^4 x + \sin^6 x\right) dx$.

   Solution:

   $$\int_0^{\frac{\pi}{2}} \left(\sin^4 x + \sin^6 x\right) dx = \int_0^{\frac{\pi}{2}} \left(\sin^4 x\right) dx + \int_0^{\frac{\pi}{2}} \left(\sin^6 x\right) dx$$

   $$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \frac{\pi}{2} \frac{1.3.5.....(n-1)}{2.4.6.....n} \text{ if } n \text{ is even and } \geq 2$$

   $$\int_0^{\frac{\pi}{2}} \sin^4 x \, dx = \frac{\pi}{2}\left(\frac{3}{8}\right) = \frac{3\pi}{16}$$

   $$\int_0^{\frac{\pi}{2}} \sin^6 x \, dx = \frac{\pi}{2}\left(\frac{3\times 5}{2\times 4\times 6}\right) = \frac{15\pi}{96}$$

   $$\int_0^{\frac{\pi}{2}} \left(\sin^4 x + \sin^6 x\right) dx = \frac{3\pi}{16} + \frac{15\pi}{96} = \frac{33\pi}{96}$$

Lecture#36

1. Evaluate the line integral $\int_c \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y,z)=xy\hat{i}+x^2z\hat{j}$ and c is given by the curve $\vec{r}(t)=t\hat{i}+2t\hat{j}+(1-t)\hat{k},\ 0\le t\le 1$.

   Solution:
   We are given that $\vec{F}(x,y,z)=xy\hat{i}+x^2z\hat{j}$ and $\vec{r}(t)=t\hat{i}+2t\hat{j}+(1-t)\hat{k}$.
   This implies
   $d\vec{r}=(\hat{i}+2\hat{j}-\hat{k})dt$.
   Also,

$$\vec{F}\left(\vec{r}(t)\right)=(2t^2)\hat{i}+t^2(1-t)\,\hat{\mathrm{j}}$$
$$=2t^2\hat{i}+(t^2-t^3)\,\hat{\mathrm{j}}$$
$$\vec{F}\cdot d\vec{r}=\left(2t^2\hat{i}+(t^2-t^3)\,\hat{\mathrm{j}}\right)\cdot\left(\hat{i}+2\hat{j}-\hat{k}\right)dt$$
$$\int_0^1 \vec{F}\cdot d\vec{r}=\int_0^1\left(2t^2+2t^2-2t^3\right)dt=\int_0^1\left(4t^2-2t^3\right)dt$$
$$=\left(\frac{4}{3}t^3-\frac{1}{2}t^4\right)\Big|_0^1=\frac{4}{3}-\frac{1}{2}=\frac{5}{6}$$

   which is the required result.

2. Evaluate the line integral $\int_c \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y,z)=yz\hat{i}+3xyz\hat{j}+2x^2y\hat{k}$ and c is the curve given by $\vec{r}(t)=\hat{i}+t\hat{j}+t^2\hat{k},\ 0\le t\le 1$.

   Solution:

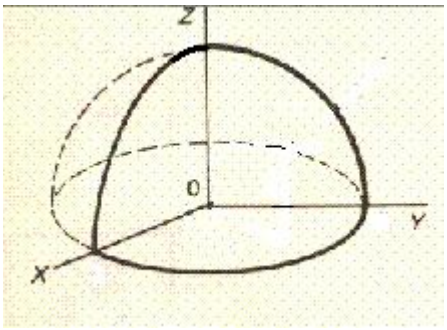
   We are given that $\vec{F}(x,y,z)=yz\hat{i}+3xyz\hat{j}+2x^2y\hat{k}$ and $\vec{r}(t)=\hat{i}+t\hat{j}+t^2\hat{k}$.
   $d\vec{r}=(\hat{j}+2t\hat{k})dt$.
   Also,

$$\vec{F}\left(\vec{r}(t)\right)=t^3\hat{i}+3t^3\hat{j}+2t\hat{k}$$
$$\vec{F}\cdot d\vec{r}=\left(t^3\hat{i}+3t^3\hat{j}+2t\hat{k}\right)\cdot\left(\hat{j}+2t\hat{k}\right)$$
$$\int_0^1 \vec{F}\cdot d\vec{r}=\int_0^1\left(3t^3+4t^2\right)dt$$
$$=\left(\frac{3t^4}{4}+\frac{4t^3}{3}\right)\Big|_0^1=\frac{3}{4}+\frac{4}{3}=\frac{25}{12}$$

Lecture#37

Consider the vector field $\vec{F} = 3xz\,\hat{i} + 2y\,\hat{j}$ over the surface S which is the portion of the sphere $x^2 + y^2 + z^2 = 9$ bounded by the planes $x = 0$, $y = 0$, $z = 0$ (as shown in the figure below).

If $\hat{n} = \frac{x}{3}\,\hat{i} + \frac{y}{3}\,\hat{j} + \frac{z}{3}\,\hat{k}$ is a unit upper normal to S. express $\int_S \vec{F} \bullet \hat{n}\, dS$ in spherica coordinates.



Note: Do not evaluate the integral.

Solution:

Put values of $\vec{F}$ and $\hat{n}$ in given integral

$$\int_S \vec{F} \bullet \hat{n}\ dS = \int_S (3xz\,\hat{i} + 2y\,\hat{j}) \bullet \left(\frac{x}{3}\,\hat{i} + \frac{y}{3}\,\hat{j} + \frac{z}{3}\,\hat{k} \right) dS$$

$$= \int_S \left(x^2 z + \frac{2}{3} y^2 \right) dS$$

To convert it to spherical polar coordinates, we have

$x = 3\sin\phi\,\cos\theta$

$y = 3\sin\phi\sin\theta$

$z = 3\cos\phi$

$dS = 9\sin\phi\, d\phi\, d\theta$

Limits are:

ϕ *ranges from* 0 *to* $\frac{\pi}{2}$

θ *ranges from* 0 *to* $\frac{\pi}{2}$

Put these values

Practice Questions

Lecture#38

Q1. Determine whether the following vector field $\vec{F}$ is conservative or not.

$a)$ $\vec{F}(x, y, z) = (2z^4 - y^3)\hat{i} + (z - 3xy^2)\hat{j} + (y + 8xz^3)\hat{k}$

b) $\vec{F}(x, y, z) = -2x\cos z\,\hat{i} + y^3\,\hat{j} + x^2 \sin z\,\hat{k}$

Q2. Let S is the surface bounded by the plane $x = 0$ $x = 1, y = 0$ $y = 2,$ $z = 0$ $z = 3$. If $\vec{F} = (2x^2y + z)\hat{i} + xz^2\,\hat{j} + (e^y - yz)\hat{k}$, use Divergence Theorem to find $\int_S \vec{F} \cdot \hat{n}\, dS$, where $\hat{n}$ denote the unit outer normal to S.

Q3. Use Stokes' Theorem to evaluate the integral $\int_s curl\,\vec{F} \bullet \hat{n}\; dS$

If $\vec{F} = z^2\hat{i} - 3xy\hat{j} + x^3y^3\hat{k}$ and surface S is the portion of paraboloid $z = 5 - x^2 - y^2$ above the plane $z = 1$. Assume that S is oriented upward.

Lecture#39

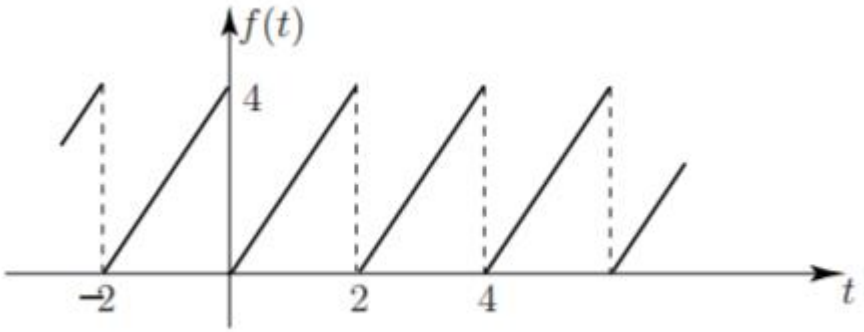
Q1. Write amplitude, period and frequency of the following:

$a)$ $y = 2\sin 3x$

b) $y = 3\sin\frac{x}{2}$

c) $y = 5\sin 8x$

Q2. Define the periodic function whose graph is shown below.



Q3. Sketch the graph of the following periodic functions showing all relevant values:

$$f(\mathrm{t}) = \begin{cases} t^2/2 & 0 < t < 4 \\ 8 & 4 < t < 6 \\ 0 & 6 < t < 8 \end{cases}$$

Q4. Evaluate

$$\int_{-\pi}^{\pi} \sin mx \sin nx \; dx = \pi \; (n = m)$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \; dx = 0 \; (n \neq m)$$

Lecture#40

Q1. Determine the Fourier co-efficient a_0 of the following function.

$$f(x) = \frac{x}{2} \qquad 0 < x < 2\pi$$

Q2**.** Determine the Fourier co-efficient a_n of the following function.

$$f(x) = \frac{x}{2} \qquad 0 < x < 2\pi$$

Q3. Determine the Fourier co-efficient b_n of the following function.

$$f(x)=\frac{x}{2} \qquad 0<x<2\pi$$

Q4. Determine the Fourier co-efficient a_0, a_n and b_n of the following function

$$f(\mathrm{x})=\begin{cases}0 & -\pi<x<0\\ x & 0<x<\pi \quad \text{and has a period } 2\pi\end{cases}$$

$$\int_S \vec{F}\bullet\hat{n}\ dS=\int_S\left(x^2z+\frac{2}{3}y^2\right)dS$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\left((3\sin\phi\ \cos\theta)^2(3\cos\phi)+\frac{2}{3}(3\sin\phi\sin\theta)^2\right)9\sin\phi\ \ d\phi\ \ d\theta$$

Practice Questions Lecture#41-43

Q1. Check whether the given function is even, odd or neither.

$f(x)=x^2e^x.$

Solution

$Let\ g(x) = \left(x^2\right)$

$h(x) = e^x$.

As $g(-x) = (-x)^2$

$= x^2$

$= g(x)$

So $g(x)$ is an even function.

$h(x)$ is neither odd nor even function. Since product of even function and "neither even nor odd function" is "neither even nor odd function"

so f(x) is neither even nor odd function.

Q2. Check whether the given function is even, odd or neither.

$f(x) = \cos^2 x \sin 5x$.

Solution

$Let\ g(x) = \cos^2 x$

$h(x) = \sin 5x$

$g(x)$ is an even function as

$g(-x) = \left(\cos(-x)\right)^2$

$= \left(\cos x\right)^2$

$= \cos^2 x = g(x)$

$h(x)$ is also an odd function as

$h(-x) = \sin 5(-x)$

$= \sin(-5x)$

$= -\sin 5x = -h(x)$

Since product of an even and and an odd function is an odd function

so $f(x)$ is an odd function.

Q3. Consider a periodic function defined by

$f(x) = x; \qquad -\pi \le x \le \pi$

Determine Fourier Co-efficient a_0, a_n and b_n.

Solution

$f(x) = x \;; \qquad -\pi \le x \le \pi$

$Since \;\; f(-x) = -x = -f(x) \;\; so \; f \; is \;\; an \, odd \;\; function.$

If *f(x)* is an odd function defined over the interval – $\pi < x < \pi$, then the Fourier series for *f(x)* contains sine terms only. So a_0 and a_n are zero.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, sin\, nx\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \, \sin nx \; dx$$

$$= \frac{2}{\pi}\left(\left| \frac{-x\cos nx}{n} \right|_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos nx \; dx \right)$$

$$= \frac{2}{n\pi}[-\pi\cos n\pi + \frac{1}{n}\sin nx|_0^{\pi}]$$

$$= \frac{-2\cos n\pi}{n} + 0 = \frac{-2\cos n\pi}{n}$$

$$= \frac{-2(-1)^n}{n}$$

Q4. Determine the Fourier coefficient a_0 for a periodic function $f(t)$ of period 4 defined by

$$f(t) = \begin{cases} (1+t) & -2 < t < 0 \\ 0 & 0 < t < 2 \end{cases}$$

Solution

Here the given function is of period 4, so T = 4 and

$$a_0 = \frac{2}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t)\,dt = \frac{2}{4}\int_{-2}^{2} f(t)\,dt = \frac{1}{2}\left[\int_{-2}^{0} f(t)\,dt + \int_{0}^{2} f(t)\,dt\right]$$

$$= \frac{1}{2}\left[\int_{-2}^{0} (1+t)\,dt + \int_{0}^{2} (0)\,dt\right] = \frac{1}{2}\left(\mathrm{t} + \frac{\mathrm{t}^2}{2}\right)_{-2}^{0} = \frac{1}{2}(2-2) = 0$$

Q5. Determine the Fourier co-efficient b_n of the periodic function

$$f(x) = 2x+2,\ 0 < x < 2$$

Solution

Here period=T=2

$$\omega = \frac{2\pi}{T} = \pi$$

$$b_n = \frac{2}{T}\int_0^T f(x)\sin n\omega x\,dx$$

$$b_n = \int_0^2 (2x+2)\sin \pi n x\,dx$$

$$= \left((2x+2)\frac{(-\cos \pi n x)}{\pi n} - 2\int \frac{(-\cos \pi n x)}{\pi n}dx\right)_0^2$$

$$= \left(-(2x+2)\frac{\cos \pi n x}{\pi n} + \frac{2(\sin \pi n x)}{\pi^2 n^2}\right)_0^2$$

$$= \left(-\frac{6}{\pi n} + \frac{2}{\pi n}\right) = \frac{-4}{\pi n}$$

Q6. Determine the Fourier co-efficient b_n of the following periodic function.

$$f(x) = x+1,\quad 0 < x < 1$$

Solution

Here period=T=1

$$\omega=\frac{2\pi}{T}=2\pi$$

$$b_n=\frac{2}{T}\int_0^T f(x)\,\sin n\omega x\;dx$$

$$b_n=2\int_0^1 (x+1)\sin 2\pi nx\;dx$$

$$=2\left((x+1)\frac{(-\cos 2\pi nx)}{2\pi n}-\int\frac{(-\cos 2\pi nx)}{2\pi n}dx\right)_0^1$$

$$=2\left(-(x+1)\frac{\cos 2\pi nx}{2\pi n}+\frac{(\sin 2\pi nx)}{4\pi^2 n^2}\right)_0^1$$

$$=2\left(\frac{-2}{2\pi n}+\frac{1}{2\pi n}\right)=\frac{-1}{\pi n}$$

Practice Questions with Solution

Lecture#44 &45

Q1. Find Laplace transform of the functions $F(t)$ if

a) $F(t)=e^{5t}$

Solution:

$$F(t)=e^{5t}$$

$$As\;L\{e^{at}\}=\frac{1}{s-a}\quad \text{where s and a are constants}$$

$$L\{e^{5t}\}=\frac{1}{s-5}$$

b) $F(t)=t^5$

Solution:

$F(t) = t^5$

$As\ L\{t^n\} = \frac{n!}{s^{n+1}}$ where s is a constant

$L\{t^5\} = \frac{5!}{s^{5+1}} = \frac{5\times4\times3\times2\times1}{s^6} = \frac{120}{s^6}$

c) F(t) = sin5t

Solution:

$F(t) = \sin 5t$

$As\ L\{\sin at\} = \frac{a}{s^2 + a^2}$ where s and a are constants

$L\{\sin 5t\} = \frac{2}{s^2 + 25}$

d) F(t) = e^{2t}cos3t

Solution:

$F(t) = e^{2t}\cos 3t$

$As\ L\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2 + b^2}$ where s,a and b are constants

$L\{e^{2t}\cos 3t\} = \frac{s-2}{(s-2)^2 + 9}$

e) F(t) = t^3 + 3t - 2

Solution:

$F(t) = t^3 + 3t - 2$

$$As\ L\{t^3+3t-2\} = L\{t^3\}+3L\{t\}-2L\{1\} = \frac{3!}{s^4}+3\,\frac{1}{s^2}-2\,\frac{1}{s}$$

Q2. Show that Laplace transform of the function

$f(t)=t$ is $\frac{1}{s^2}$ where s is a constant for the integration and s > 0.

Solution:

As we know

$$L[f(t)] = \int_0^{+\infty} e^{-st}\ f(t)\,dt$$

$$So\ for\ f(t)=t$$

$$L[\mathrm{t}] = \int_0^{+\infty} e^{-st}t\ \ dt$$

$$= \lim_{l\to+\infty}\int_0^{l} e^{-st}t\ \ dt$$

$$= \lim_{l\to+\infty}\left\{\left|t\frac{e^{-st}}{s}\right|_0^{l} + \int_0^{l} 1.\frac{e^{-st}}{s}dt\right\}$$

$$= \lim_{l\to+\infty}\left\{\left|t\frac{e^{-st}}{s}\right|_0^{l} - \left|\frac{e^{-st}}{s^2}\right|_0^{l}\right\}$$

In order for this limit to exist, we aga in must insist that $s \neq 0$ and that $s > 0$ so that e^{-sl} has a limit (of zero). We obtain

$$= -\frac{1}{s}\lim_{l\to+\infty}(l\ e^{-sl}-0) - \frac{1}{s^2}\lim_{l\to+\infty}(e^{-sl}-1)$$

$$= \frac{1}{s^2}\ ; \qquad s>0$$

Q3. Find Laplace transform of the function $F(t)$ if

$F(t) = e^{2t} \cos t$

Solution:

Since

$$L\{\cos t\} = \frac{s}{s^2+1}$$

So by First Shift Theorem

$$L\{e^{-(-2t)} \cos 2t\} = \frac{s-2}{(s-2)^2+1}$$

$$= \frac{s-2}{(s^2+4-4s)+1}$$

$$= \frac{s-2}{s^2-4s+5}$$

Q4. Determined:

a) $\mathbf{L^{-1}\left\{\frac{3}{s^2+4}\right\}}$

Solution:

$$L^{-1}\left\{\frac{3}{s^2+4}\right\} = L^{-1}\left\{\frac{3}{2}\frac{3}{s^2+2^2}\right\} = \frac{3}{2}L^{-1}\left\{\frac{3}{2}\frac{3}{s^2+2^2}\right\} = \frac{3}{2}\sin 2t$$

b) $\mathbf{L^{-1}\left\{\frac{2}{(s+5)^4}\right\}}$

Solution:

$$\mathbf{L^{-1}}\left\{\frac{2}{(\mathbf{s}+5)^4}\right\} = \mathrm{L}^{-1}\left\{\frac{1}{3}\frac{6}{(\mathbf{s}+5)^4}\right\} = \frac{1}{3}\mathrm{L}^{-1}\left\{\frac{3!}{(\mathbf{s}+5)^4}\right\} = \frac{1}{3}\mathrm{e}^{-5t}\mathrm{L}^{-1}\left\{\frac{3!}{\mathbf{s}^4}\right\} = \frac{1}{3}\mathrm{e}^{-5t}t^3$$

Q5. Find Laplace transform of the function $F(t)$ if

$F(t) = e^{3t} \sin 4t$

Solution:

Since

$$L\{\sin 4t\} = \frac{4}{s^2+16}$$

So by First Shift Theorem

$$L\{e^{-(-3t)} \sin 4t\} = \frac{4}{(s-3)^2+16}$$

$$= \frac{4}{(s^2+9-6s)+16}$$

$$= \frac{4}{s^2-6s+25}$$