

ORANGE MONKEY

MIDTERM MCQS

Mth603

1. Which method requires a derivative of the solution.

Regular-Falsi method

2. In Gauss-Seidel method, each equation of the system is solved for the unknown with _____ coefficient, in terms of remaining unknowns.

Largest.....confirm

Solution

At first, we transfer all the terms to the right-hand side and reorder the equations, so that the largest coefficient in the equations appear on the diagonal.

3. The convergence of which of the following method is sensitive to starting value

Newton-Raphson Method.....confirm

Secant Method

The secant method is modified form of Newton-Raphson method. If in Newton-Raphson method, we replace the derivative $f'(x_n)$ by the difference ratio, i.e.,

$$f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Where x_n and x_{n-1} are two approximations of the root we get

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \\ &= \frac{x_n f(x_n) - x_n f(x_{n-1}) - f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \\ &= \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \end{aligned}$$

Provided $f(x_n) \neq f(x_{n-1})$

This method requires two starting values x_0 and x_1 ; values at both these points are calculated which gives two points on the curve the next point on the curve are obtained by using the derived formula. We continue this procedure until we get the root of required accuracy.

Geometrical Interpretation

4. The number of significant digits in the number 608.030060 is

6

5. For a system of linear equations, the corresponding coefficient matrix has the value of determinant; $|A| = -3$, then which of the following is true?

The system has unique solution

6. The number system that has a base 8 is called _____ system

8

7. If there are three equations in two variables then which of the following is true?

All above possibilities exist depends on the coefficients of variables

8. The root of the equation $3x - e^x = 0$ is bounded in the interval.

[0,1]

9. In _____ method matrix $[A]$ of the system of equations is decomposed into the product of two matrices $[L]$ and $[U]$ where $[L]$ is a lower-triangular matrix and $[U]$ is an upper-triangular matrix with 1's on its main diagonal.

Crout's Reduction.....confirm

Crout's Reduction Method

Here the coefficient matrix $[A]$ of the system of equations is decomposed into the product of two matrices $[L]$ and $[U]$, where $[L]$ is a lower-triangular matrix and $[U]$ is an upper-triangular matrix with 1's on its main diagonal.

For further illustration, consider a general matrix in the form

10. Which method requires three starting points.

Muller method.....confirm

Muller's Method

Muller's method is a generalization of the secant method, in the sense that it does not require the derivative of the function. It is an iterative method that requires three starting points $(p_0, f(p_0))$, $(p_1, f(p_1))$, and $(p_2, f(p_2))$. A parabola is constructed that passes through the three points; then the quadratic formula is used to find a root of the quadratic for the next approximation. It has been proved that near a simple root

11. For the system of equations $x=2$, $y=3$. The inverse of the matrix associated with its coefficients is _____.

Non-identity

12. While using the Gauss-Seidel Method for finding the solution of the system of

$$3x + y + z = 11$$

$$2x + 5y + z = 16$$

equation, the following system $z + y + 5z = 4$ can be rewritten as

$$x = \frac{1}{3}(11 - y - z)$$

$$y = \frac{1}{5}(16 - 2x - z)$$

$$z = \frac{1}{5}(4 - x - y)$$

13. While using the relaxation method for finding the solution of the below given system

$$6x_1 - 2x_2 + 3x_3 = 1$$

$$-2x_1 + 7x_2 + 2x_3 = 5$$

which of the following increment will be introduced $x_1 + x_2 - 5x_3 = -13$

$$dx_3 = \frac{R_3}{a_{33}}$$

14. Gauss-Seidel method is also known as method of

Successive displacement

15. For a system of linear equations, the corresponding coefficient matrix has the value of determinant $|A| = -3$ then which of the following is true?

The system has unique solution

16. If the product of two matrices is an identity matrices that is $AB = I$ then which of the following is true?

A is inverse of B

17. Which of the following is equivalent form of the system of equations in matrix form $AX = B$?

$$X = (\text{Inverse of } A)B$$

18. If the determinant of a matrix A is equal to zero then the system of equations will have.....

No solutions or infinitely many solution

19. While solving the system of linear equation $x-y=2$, $-x+y=3$ by Jacobi's method, if $(0,0)$ be its first approximate solution, then which of the following is second approximate solution?

$(2,3)$

20. While solving a system of linear equations by Gauss Jordan Method in which of the following case the partial pivoting is essential?

When pivots are zero

21. If the pivot element happens to be zero, then the i -th column elements are searched for the numerically _____ element

largest

22. For two matrices A and B, such that "A=Inverse of B", then which of the following is true?

All options are correct

23. Which of the following rearrangement make strictly diagonal dominant, the system of linear equations; $x-3y+z=-2$, $-6x+4y+11z=1$, $5x-2y-2z=9$?

$5x-2y-2z=9$, $x-3y+z=-2$, $-6x+4y+11z=1$

24. In Jacobi's method, We assume that the _____ elements does not vanish.

Diagonal

25. While solving by Gauss-Seidel method which of the following is the first iterative solution for the system $x-2y=1$ $x+4y=4$?

$(1, 0.75)$

26. While solving a system of three linear equations in three variables by Jacobi's method, the first approximate solution is necessary to be taken as $(0,0,0)$

True

27. Gauss-Seidel method is similar to _____.

Jacobi's method

28. If $A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ then by using Gaussian Elimination method the value of A^{-1} will be

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 2 & 3 & -\frac{1}{2} \\ 1 & 4 & 2 \\ 5 & -\frac{2}{3} & \frac{1}{3} \\ 6 & 3 & 3 \end{bmatrix}$$

29. If $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 4 & -1 \\ 3 & 1 & 3 \end{bmatrix}$ then by using Gaussian Elimination method the value of A^{-1} will be

$$\begin{bmatrix} \frac{11}{6} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & 0 & 0 \\ -\frac{11}{6} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

30. The Gaussian elimination method fails if any one of the pivot element becomes zero

31. The 2nd row of the augmented matrix of the system of linear equations is:

$$2x + z = 4$$

$$x - y + z = -3$$

$$-y + z = 5$$

1, -1, 1 and -3

32. The 3rd row of the augmented matrix of the system of linear equations is:

$$2x + z = 4$$

$$x - y + z = -3$$

$$-y + z = -5$$

0, -1, 1 and -5

33. The first row of the augmented matrix of the system of linear equations is:

$$2x + z = 4$$

$$x - y + z = -3$$

$$-y + z = 5$$

2, 0, 1 and 4

34. The first row of the augmented matrix of the system of linear equations is:

$$2x + z = 4$$

$$x - y + z = -3$$

$$-y + z = -5$$

2, 0, 1 and 4

35. The system of linear equations; $2x+3y=6$, $x+4y=5$ is not strictly diagonal dominant as _____.

$$|2| < |3|$$

36. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 3 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} \frac{7}{3} & \frac{1}{3} & -\frac{2}{5} \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ \frac{11}{10} & -\frac{1}{3} & -\frac{1}{10} \end{bmatrix}$ then the value of AA^{-1} will be

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

37. In Jacobi's Method, We assume that the _____ elements does not vanish.

Diagonal

38. While using the Gauss Seidel Method for finding the solution of the system of

$$3x + y + z = 11$$

$$2x + 5y - z = 16$$

equation, the following system $x + y + 5z = 4$ can be written as

$$\left(\frac{11}{3}, \frac{16}{5}, -\frac{4}{5}\right)$$

39. While using the Gauss Seidel Method for finding the solution of the system of

$$2x + 2y + z = 3$$

$$x + 3y + z = 2$$

equation, the following system $x + y + z = 2$ can be written as

$$\left(\frac{3}{2}, \frac{2}{3}, 2\right)$$

40. While using the Gauss Seidel Method for finding the solution of the system of

$$2x + 2y + z = 3$$

$$x + 3y + z = 2$$

equation, the following system $x + y + z = 2$ can be written as

$$x = \frac{3}{2} - y - \frac{z}{2}$$

$$y = \frac{2}{3} - \frac{x}{3} - \frac{z}{3}$$

$$z = 2 - x - y$$

41. While using the Gauss Seidel Method for finding the solution of the system of

$$x + 2y + 2z = 3$$

$$x + 3y + 3z = 2$$

equation, the following system $x + y + 5z = 2$ can be written as

$$x = 3 - 2y - 2z$$

$$y = \frac{2}{3} - \frac{x}{3} - z$$

$$z = \frac{2}{5} - \frac{x}{5} - \frac{y}{5}$$

42. While using the relaxation method for finding the solution of the following system

$$x + 2y + 2z = 3$$

$$x + 3y + 3z = 2$$

$x + y + 5z = 2$ with the initial guess (0,0,0) the next iteration would be

$$(3, 2/3, 2/5)$$

43. While using the relaxation method for finding the solution of the following system

$$8x_1 + 3x_2 - 2x_3 = 5$$

$$4x_1 + 7x_2 + 2x_3 = 3$$

$3x_1 + 5x_2 + 9x_3 = 2$ with the initial vector (0,0,0) the residuals would be

$$R_1=5, R_2=9, R_3=2$$

44. While using the relaxation method for finding the solution of the following system

$$11x_1 + x_2 - x_3 = 8$$

$$x_1 + 8x_2 + 5x_3 = 9$$

$x_1 + x_2 + 9x_3 = 7$ with the initial vector (0,0,0) the residuals would be

$$R_1=8, R_2=9, R_3=7$$

45. While using the relaxation method for finding the solution of the following system

$$5x_1 + x_2 - 3x_3 = 5$$

$$x_1 + 5x_2 + 2x_3 = 7$$

$x_1 + 2x_2 + 9x_3 = 3$ with the initial vector (0,0,0) the residuals would be

$R_1=5, R_2=7$ and $R_3=3$

46. While using the relaxation method for finding the solution of the following system

$$8x_1 + 3x_2 - 2x_3 = 5$$

$$4x_1 + 7x_2 + 2x_3 = 9$$

$$3x_1 + 5x_2 + 9x_3 = 2$$
 with the initial vector $(0,0,0)$ the residuals would be

$R_1=5, R_2=9, R_3=2$

47. Gauss – Jordan Method is similar to _____.

Gaussian elimination method

48. For the system; $2x+3y = 1$, $3x + 2y = -4$, if the iterative solution is $(0,0)$ and 'dxi = 2' is the increment in 'y' then which of the following will be taken as next iterative solution?

$(0,2)$

49. While solving a system of linear equations, which of the following approach is economical for the computer memory?

Iterative.....confirm

(many elements are zero), iterative method has definite advantage over direct methods in respect of economy of computer memory

50. Let A be an $n \times n$ matrix. The number x is an eigenvalue of A if there exists a non-zero vector such that $Av=xv$

True

51. An augmented matrix may also be used to find the inverse of a matrix by combining it with the _____ matrix

Identity.....confirm

Instead, we will augment the original matrix with the identity matrix and use row operations to obtain the inverse.



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52. In Gauss-Jacobi's method, the corresponding elements of $x_1^{(r+1)}$ replaces those of $x_1^{(r)}$ as soon as they become available.

True

53. If $n \times n$ matrix A and B are similar, then they have the _____ eigenvalues (with the same multiplicities).

Same.....confirm

Question # 2 of 10 (Total Marks: 1)

If $n \times n$ matrices A and B are similar, then they have the different eigenvalues (with the same multiplicities).

1. True

False

54. $u^1 = \begin{pmatrix} 7 \\ 12 \\ 14 \end{pmatrix}$ Will be in normalized form as

$$u^1 = \frac{1}{14} \begin{pmatrix} 7/14 \\ 12/14 \\ 1 \end{pmatrix}$$

55. While using Gaussian Elimination method, the following augmented matrix

$$[A][I] = \begin{bmatrix} 1 & \frac{3}{4} & 0 & \frac{11}{40} & \frac{1}{5} & -\frac{1}{40} \\ 0 & 1 & 0 & -\frac{3}{2} & 0 & 1 \\ 0 & 0 & 1 & \frac{11}{40} & -\frac{1}{5} & -\frac{1}{10} \end{bmatrix}$$

will reduce in identity matrix by performing.

$$R_1 - \frac{3}{4}R_2$$

56. While using relaxation method, which of the following is increment 'dxi' corresponding to the largest residual for 1st iteration on the system $2x+3y=1$, $3x+2y=-4$.

-- 4.....confirm

57. Gauss elimination and Gauss-Jordan methods are popular among many methods for finding the _____ of a matrix

Inverse.....confirm

Every square non-singular matrix will have an inverse.

Gauss elimination and Gauss-Jordan methods are popular among many methods available for finding the inverse of a matrix.

58. Let $[A]$ be a 3×3 real symmetric matrix with $|a_{23}|$ be the numerically largest off-diagonal element then using Jacobi's method the value of θ can be found by

$$\tan 2\theta = \frac{2a_{23}}{a_{22} - a_{33}}$$

59. While using Gaussian-Jordan Method, we have the following augmented matrix

$$[A][I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ 0 & -1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array} \right]$$

Then the value of A^{-1} will be

$$\left[\begin{array}{ccc} \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array} \right]$$

60. Let $[A]$ be a 3×3 real symmetric matrix with $|a_{12}|$ be numerically the largest off-diagonal element of A , then we can construct orthogonal matrix S_q by Jacobi's method as

$$\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

61. Let $[A]$ be a 3×3 real symmetric matrix with $|a_{13}|$ be numerically the largest off-diagonal element of A , then we can construct orthogonal matrix S_q by Jacobi's method as

$$\begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix}$$

62. Which of the following system of equation is diagonally dominant?

$$11x + y - z = 4$$

$$x + 8y + 5z = 2$$

$$x + y + 9z = 3$$

63. Which of the following system of equation is diagonally dominant?

$$8x + y + z = 5$$

$$2x + 13y + 5z = 7$$

$$x + 2y + 9z = 3$$

64. While using Jacobi method for the matrix

$$A = \begin{bmatrix} 1 & \sqrt{3} & \sqrt{2} \\ \sqrt{3} & 3 & 2 \\ \sqrt{2} & 2 & 1 \end{bmatrix}$$

the value of theta 0 can be found as

$$\tan 2\theta = \frac{2a_{23}}{a_{22} - a_{33}}$$

65. While using Jacobi method for the matrix

$$A = \begin{bmatrix} 1 & 1/4 & 1/3 \\ 1/4 & 1/3 & 1/2 \\ 1/3 & 1/2 & 1/5 \end{bmatrix}$$

The value of theta – can be found as

$$\tan 2\theta = \frac{2a_{23}}{a_{23} - a_{33}}$$

66. Which of the following systems of linear equations has a strictly diagonally dominant coefficient matrix?

$$\begin{aligned} 6x_1 - 2x_2 + 3x_3 &= 1 \\ -2x_1 + 7x_2 + 2x_3 &= 5 \\ x_1 + x_2 - 2x_3 &= -13 \end{aligned}$$

67. Which of the following systems of linear equations has a strictly diagonally dominant coefficient matrix?

$$\begin{aligned} 9x_1 + 5x_2 - 3x_3 &= 12 \\ -x_1 + 12x_2 + 5x_3 &= 8 \\ 2x_1 - 4x_2 + 7x_3 &= -15 \end{aligned}$$

68. We have the largest eigen value and the corresponding eigenvector as

$$u^3 = 3.17890 \begin{pmatrix} 1.0 \\ 1.231926 \\ 1.421389 \end{pmatrix}$$

$$\lambda = 3.17890, (X) = \begin{pmatrix} 1.0 \\ 1.231926 \\ 1.421389 \end{pmatrix}$$

69. The linear equation $0x+0y=2$ has _____ solution/solutions

No solution

70. The linear equation $x+y=1$ has _____ solution/solutions

Infinite many

71. In _____ method, a system is reduced to an equivalent diagonal form using elementary transformations.

Gaussian elimination.....confirm

Solution of Linear System of Equations and Matrix Inversion Gauss-Jordan

Elimination Method

This method is a variation of Gaussian elimination method. In this method, the elements above and below the diagonal are simultaneously made zero. That is a given system is reduced to an equivalent diagonal form using elementary transformations. Then the solution of the resulting diagonal system is obtained. Sometimes, we normalize the pivot row with respect to the pivot element, before elimination. Partial pivoting is also used whenever the pivot element becomes zero.

72. In _____ method the elements above and below the diagonal are simultaneously made zero.

Gauss-Jordan Elimination.....confirm

Solution of Linear System of Equations and Matrix Inversion Gauss-Jordan

Elimination Method

This method is a variation of Gaussian elimination method. In this method, the elements above and below the diagonal are simultaneously made zero. That is a given system is reduced to an equivalent diagonal form using elementary transformations. Then the solution of the resulting diagonal system is obtained. Sometimes, we normalize the pivot row with respect to the pivot element, before elimination. Partial pivoting is also used whenever the pivot element becomes zero.

73. In full pivoting we interchange rows and columns such that the _____ elements in the matrix of the variables also get changed.

Largest.....confirm

In full pivoting (or complete pivoting), we interchange rows and columns, such that the largest element in the matrix of the variables also get changed. Full pivoting, in fact, is more complicated than the partial pivoting. Partial pivoting is preferred for hand computation.

Numerical Analysis

MTH603



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Knowledge beyond the boundaries

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Numerical Analysis

Course Contents

Solution of Non Linear Equations
 Solution of Linear System of Equations
 Approximation of Eigen Values
 Interpolation and Polynomial Approximation
 Numerical Differentiation
 Numerical Integration
 Numerical Solution of Ordinary Differential Equations

Introduction

We begin this chapter with some of the basic concept of representation of numbers on computers and errors introduced during computation. Problem solving using computers and the steps involved are also discussed in brief.

Number (s) System (s)

In our daily life, we use numbers based on the decimal system. In this system, we use ten symbols 0, 1, ..., 9 and the number 10 is called the base of the system.

Thus, when a base N is given, we need N different symbols 0, 1, 2, ..., $(N - 1)$ to represent an arbitrary number.

The number systems commonly used in computers are

Base, N	Number
2	Binary
8	Octal
10	Decimal
16	Hexadecimal

An arbitrary real number, a can be written as

$$a = a_m N^m + a_{m-1} N^{m-1} + \cdots + a_1 N^1 + a_0 + a_{-1} N^{-1} + \cdots + a_{-m} N^{-m}$$

In binary system, it has the form,

$$a = a_m 2^m + a_{m-1} 2^{m-1} + \cdots + a_1 2^1 + a_0 + a_{-1} 2^{-1} + \cdots + a_{-m} 2^{-m}$$

The decimal number 1729 is represented and calculated

$$(1729)_{10} = 1 \times 10^3 + 7 \times 10^2 + 2 \times 10^1 + 9 \times 10^0$$

While the decimal equivalent of binary number 10011001 is

$$\begin{aligned} & 1 \times 2^0 + 0 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3} + 1 \times 2^{-4} + 0 \times 2^{-5} + 0 \times 2^{-6} + 1 \times 2^{-7} \\ &= 1 + \frac{1}{8} + \frac{1}{16} + \frac{1}{128} \\ &= (1.1953125)_{10} \end{aligned}$$

Electronic computers use binary system whose base is 2. The two symbols used in this system are 0 and 1, which are called *binary digits* or simply *bits*.

The internal representation of any data within a computer is in binary form. However, we prefer data input and output of numerical results in decimal system. Within the computer, the arithmetic is carried out in binary form.

Conversion of decimal number 47 into its binary equivalent

Sol.

2	47	Remainder
2	23	1
2	11	1
2	5	1
2	2	1
2	1	0
	0	1 Most significant bit

$$(47)_{10} = (101111)_2$$

Binary equivalent of the decimal fraction 0.7625

Sol.

		Product	Integer
0.7625	x2	1.5250	1
0.5250	x2	1.0500	1
0.05	x2	0.1	0
0.1	x2	0.2	0
0.2	x2	0.4	0
0.4	x2	0.8	0
0.8	x2	1.6	1
0.6	x2	1.2	1
0.2	x2	0.4	0

$$(0.7625)_{10} = (0.11\dots 11(0011))_2$$

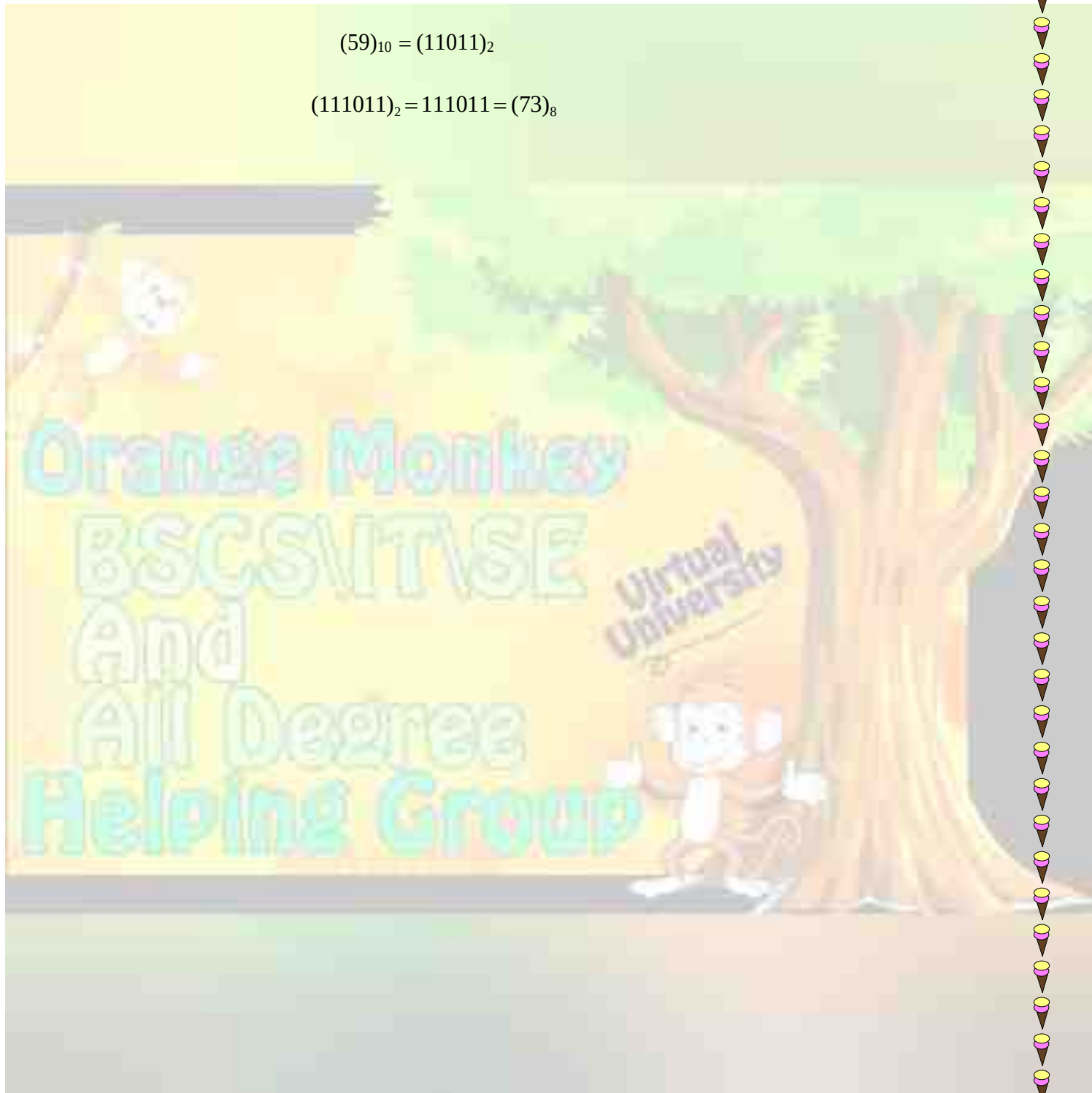
Conversion $(59)_{10}$ into binary and then into octal.

Sol.

$$\begin{array}{r}
 2 \overline{) 29} \quad 1 \\
 \underline{20} \\
 9 \quad 1 \\
 \underline{8} \\
 1 \quad 0 \\
 \underline{0} \\
 1 \quad 1 \\
 \underline{2} \\
 0 \quad 1
 \end{array}$$

$$(59)_{10} = (11011)_2$$

$$(111011)_2 = 111011 = (73)_8$$



Errors in Computations

Numerically, computed solutions are subject to certain errors. It may be fruitful to identify the error sources and their growth while classifying the errors in numerical computation. These are

Inherent errors,

Local round-off errors

Local truncation errors

Inherent errors

It is that quantity of error which is present in the statement of the problem itself, before finding its solution. It arises due to the simplified assumptions made in the mathematical modeling of a problem. It can also arise when the data is obtained from certain physical measurements of the parameters of the problem.

Local round-off errors

Every computer has a finite word length and therefore it is possible to store only a fixed number of digits of a given input number. Since computers store information in binary form, storing an exact decimal number in its binary form into the computer memory gives an error. This error is computer dependent.

At the end of computation of a particular problem, the final results in the computer, which is obviously in binary form, should be converted into decimal form-a form understandable to the user-before their print out. Therefore, an additional error is committed at this stage too.

This error is called *local round-off error*.

$$(0.7625)_{10} = (0.110000110011)_2$$

If a particular computer system has a word length of 12 bits only, then the decimal number 0.7625 is stored in the computer memory in binary form as 0.110000110011. However, it is equivalent to 0.76245.

Thus, in storing the number 0.7625, we have committed an error equal to 0.00005, which is the round-off error; inherent with the computer system considered.

Thus, we define the *error* as

$$\text{Error} = \text{True value} - \text{Computed value}$$

Absolute error, denoted by $|\text{Error}|$,

While, the *relative error* is defined as

$$\text{Relative error} = \frac{|\text{Error}|}{|\text{True value}|}$$

Local truncation error

It is generally easier to expand a function into a power series using Taylor series expansion and evaluate it by retaining the first few terms. For example, we may approximate the function $f(x) = \cos x$ by the series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

If we use only the first three terms to compute $\cos x$ for a given x , we get an approximate answer. Here, the error is due to truncating the series. Suppose, we retain the first n terms, the truncation error (TE) is given by

$$TE \leq \frac{x^{2n+2}}{(2n+2)!}$$

The TE is independent of the computer used.

If we wish to compute $\cos x$ for accurate with five significant digits, the question is, how many terms in the expansion are to be included? In this situation

$$\frac{x^{2n+2}}{(2n+2)!} < .5 \times 10^{-5} = 5 \times 10^{-6}$$

Taking logarithm on both sides, we get

$$(2n+2) \log x - \log[(2n+2)!] \\ < \log_{10} 5 - 6 \log_{10} 10 = 0.699 - 6 = -5.3$$

or

$$\log[(2n+2)!] - (2n+2) \log x > 5.3$$

We can observe that, the above inequality is satisfied for $n = 7$. Hence, seven terms in the expansion are required to get the value of $\cos x$, with the prescribed accuracy

The truncation error is given by

$$TE \leq \frac{x^{16}}{16!}$$



Polynomial

An expression of the form $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n$ where n is a

positive integer and $a_0, a_1, a_2, \dots, a_n$ are real constants, such type of expression is called an n th degree polynomial in x if $a_0 \neq 0$

Algebraic equation:

An equation $f(x)=0$ is said to be the algebraic equation in x if it is purely a polynomial in x .

For example

$x^5 + x^4 + 3x^2 + x - 6 = 0$ It is a fifth order polynomial and so this equation is an algebraic equation.

$$x^3 - 6 = 0$$

$$x^6 - 7x = 0$$

$$y^4 - 4y^3 + 3y^2 - y - 2 = 0 \text{ polynomial in } y$$

$$t^4 - 6t^2 - 21 = 0 \text{ polynomial in } t$$

These all are the examples of the polynomial or algebraic equations.

Some facts

1. Every equation of the form $f(x)=0$ has at least one root, it may be real or complex.
2. Every polynomial of n th degree has n and only n roots.
3. If $f(x)=0$ is an equation of odd degree, then it has at least one real root whose sign is opposite to that of last term.
4. If $f(x)=0$ is an equation of even degree whose last term is negative then it has at least one positive and at least one negative root.

Transcendental equation

An equation is said to be transcendental equation if it has logarithmic, trigonometric and exponential function or combination of all these three.

For example

$e^x - 5x - 3 = 0$ it is a transcendental equation as it has an exponential function

$$e^x - \sin x = 0$$

$$\ln x - \sin x = 0$$

$$2 \sec^2 x - \tan x - e^x = 0$$

These all are the examples of transcendental equation.

Root of an equation

For an equation $f(x)=0$ to find the solution we find such value which satisfy the equation $f(x)=0$, these values are known as the roots of the equation.

A value a is known as the root of an equation $f(x)=0$ if and only if $f(a)=0$.

Properties of an Algebraic equation

1. Complex roots occur in the pairs. That is, If $(a+ib)$ is a root of $f(x)=0$ then $(a-ib)$ is also a root of the equation
2. if $x=a$ is a root of the equation $f(x)=0$ a polynomial of n th degree, then $(x-a)$ is a factor of $f(x)$ and by dividing $f(x)$ by $(x-a)$ we get a polynomial of degree $n-1$.

Descartes rule of signs

This rule shows the relation ship between the signs of coefficients of an equation and its roots.

“The number of positive roots of an algebraic equation $f(x)=0$ with real coefficients can not exceed the number of changes in the signs of the coefficients in the polynomial $f(x)=0$. similarly the number of negative roots of the equation can not exceed the number of changes in the sign of coefficients of $f(-x)=0$ ”

Consider the equation $x^3 - 3x^2 + 4x - 5 = 0$ here it is an equation of degree three and there are three changes in the signs

First +ve to -ve second -ve to +ve and third +ve to -ve so the tree roots will be positive

Now $f(-x) = -x^3 - 3x^2 - 4x - 5$ so there is no change of sign so there will be no negative root of this equation.

Intermediate value property

If $f(x)$ is a real valued continuous function in the closed interval $a \leq x \leq b$ if $f(a)$ and $f(b)$ have opposite signs once; that is $f(x)=0$ has at least one root β such that $a \leq \beta \leq b$

Simply

If $f(x)=0$ is a polynomial equation and if $f(a)$ and $f(b)$ are of different signs, then $f(x)=0$ must have at least one real root between a and b .

Numerical methods for solving either algebraic or transcendental equation are classified into two groups

Direct methods

Those methods which do not require any information about the initial approximation of root to start the solution are known as direct methods.

The examples of direct methods are Graefee root squaring method, Gauss elimination method and Gauss Jordan method. All these methods do not require any type of initial approximation.

Iterative methods

These methods require an initial approximation to start.

Bisection method, Newton raphson method, secant method, jacobie method are all examples of iterative methods.

How to get an initial approximation?

The initial approximation may be found by two methods either by graphical method or analytical method

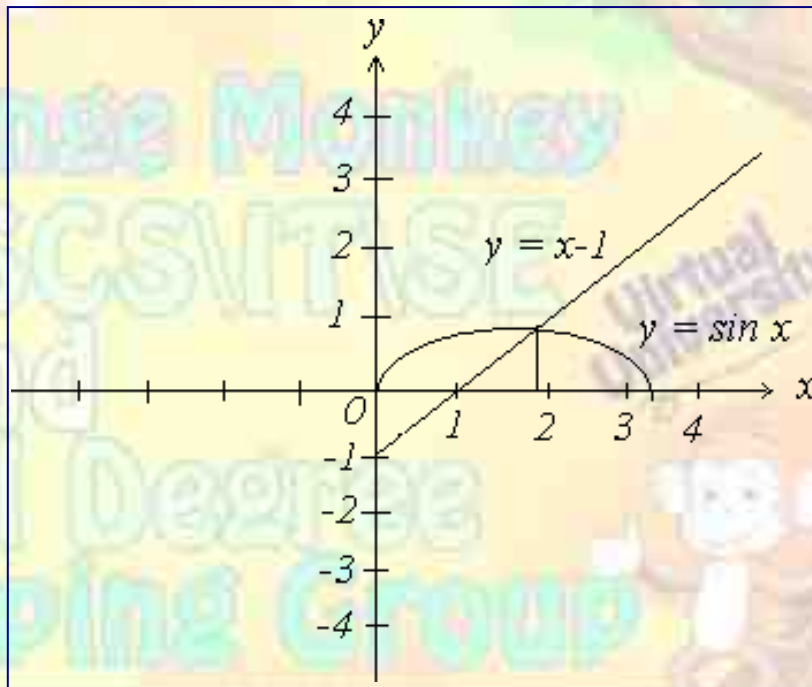
Graphical method

The equation $f(x)=0$ can be rewritten as $f_1(x) = f_2(x)$ and initial approximation of $f(x)$ may be taken as the abscissa of the point of intersection of graphs of

$y = f_1(x)$ and $y = f_2(x)$

for example $f(x) = x - \sin x - 1 = 0$

so this may be written as $x - 1 = \sin x$ Now we shall draw the graphs of $y = x - 1$ and $y = \sin x$



Here both the graphs cut each other at 1.9 so the initial approximation should be taken as 1.9

Analytical method

This method is based on the intermediate value property in this we locate two values a and b such that $f(a)$ and $f(b)$ have opposite signs then we use the fact that the root lies

between both these points ,this method can be used for both transcendental and algebraic equations.

Consider the equation

$$f(x) = 3x - \sqrt{1 + \sin x} = 0$$

$$f(1) = 3 - \sqrt{1 + \sin(1 \times \frac{180}{\pi})} = 3 - \sqrt{1 + 0.84147} = 1.64299$$

Here $f(0)$ and $f(1)$ are of opposite signs making use of intermediate value property we infer that one root lies between 0 and 1 .

So in analytical method we must always start with an initial interval (a,b) so that $f(a)$ and $f(b)$ have opposite signs.

Bisection method (Bolzano)

Suppose you have to locate the root of the equation $f(x)=0$ in an interval say (x_0, x_1) , let $f(x_0)$ and $f(x_1)$ are of opposite signs such that $f(x_0)f(x_1) < 0$

Then the graph of the function crossed the x -axis between x_0 and x_1 which exists the existence of at least one root in the interval (x_0, x_1) .

The desired root is approximately defined by the mid point $x_2 = \frac{x_0 + x_1}{2}$ if $f(x_2) = 0$ then

x_2 is the root of the equation otherwise the root lies either between x_0 and x_2 or x_1 and x_2

Now we define the next approximation by $x_3 = \frac{x_0 + x_2}{2}$ provided $f(x_0)f(x_2) < 0$ then

root may be found between x_0 and x_2

If provided $f(x_1)f(x_2) < 0$ then root may be found between x_1 and x_2 by $x_3 = \frac{x_1 + x_2}{2}$

Thus at each step we find the desired root to the required accuracy or narrow the range to half the previous interval.

This process of halving the intervals is continued in order to get smaller and smaller interval within which the desired root lies. Continuation of this process eventually gives us the required root.

NOTE: In numerical analysis all the calculation are carried out in radians mode and the value of pi is supposed to be 3.14

Example

Solve $x^3 - 9x + 1 = 0$ for the root between $x=2$ and $x=4$ by bisection method

Solution:

Here we are given the interval $(2,4)$ so we need not to carry out intermediate value property to locate initial approximation.

Here

$$f(x) = x^3 - 9x + 1 = 0$$

$$\text{now } f(2) = 2^3 - 9(2) + 1 = 8 - 18 + 1 = -9$$

$$f(4) = 4^3 - 9(4) + 1 = 64 - 36 + 1 = 29$$

here $f(2) f(4) < 0$ so root lies between 2 and 4

$$x_0 = 2 \quad x_1 = 4$$

$$x_2 = \frac{2+4}{2} = 3$$

$$f(3) = 3^3 - 9(3) + 1 = 27 - 27 + 1 = 1$$

here $f(2) f(3) < 0$ so the root lies between 2 and 3

$$x_3 = \frac{2+3}{2} = 2.5$$

$$f(2.5) = 2.5^3 - 9(2.5) + 1 = 15.625 - 22.5 + 1 = -5.875 < 0$$

so the root lies between 2.5 and 3 as $f(2.5) f(3) < 0$

$$x_4 = \frac{2.5+3}{2} = 2.75$$

now

similarly $x_5 = 2.875$ and $x_6 = 2.9375$ and the process is continued

until the desired accuracy is obtained.

n	x_n	$f(x_n)$
2	3	1.0
3	2.5	-5.875
4	2.75	-2.9531
5	2.875	-1.1113
6	2.9375	-0.0901

When to stop the process of iteration?

Here in the solved example neither accuracy is mentioned and nor number of iteration is mentioned by when you are provided with the number of iteration then you will carry those number of iteration and will stop but second case is accuracy like you are asked to

find root with an accuracy of 10^{-3} then you will check the accuracy by subtracting two consecutive root like 2.135648 and 2.135769

$$2.135769 - 2.135648 = 0.000121$$

So the accuracy of 10^{-3} is achieved so you will stop here.

Example:

Carry out the five iterations for the function $f(x) = 2x \cos(2x) - (x+1)^2$

Note: All the calculations should be done in the radians.

Solution:

$$f(x) = 2x \cos(2x) - (x+1)^2$$

$$f(-1) = 2(-1) \cos(-2) - (-1+1)^2 = -2(-0.4161) = +0.8322 > 0$$

$$f(0) = 2(0) \cos(0) - (0+1)^2 = -1 = -1 < 0$$

so the root lies between 0 and -1 as $f(0)f(-1) < 0$

$$x_2 = \frac{0-1}{2} = -0.5$$

$$f(-0.5) = 2(-0.5) \cos(-1) - (-0.5+1)^2 = -0.5403 - 0.25 = -0.7903 < 0$$

so root lies between -1 and -0.5 as $f(-1)f(-0.5) > 0$

$$x_3 = \frac{-0.5-1}{2} = -0.75$$

$$f(-0.75) = 2(-0.75) \cos(-1.5) - (-0.75+1)^2 = -0.106 - 0.0625 = -0.1686 < 0$$

so root lies between -1 and -0.75 as $f(-1)f(-0.75) > 0$

$$x_4 = \frac{-0.75-1}{2} = -0.875$$

$$f(-0.875) = 2(-0.875) \cos(-1.75) - (-0.875+1)^2 = 0.3119 - 0.015625 = 0.296275 > 0$$

so root lies between -0.875 and -0.75 as $f(-0.75)f(-0.875) < 0$

$$x_5 = \frac{-0.75-0.875}{2} = -0.8125$$

$$f(-0.8125) = 2(-0.8125) \cos(-1.625) - (-0.8125+1)^2 = 0.0880 - 0.0351 = 0.052970 > 0$$

so root lies between -0.8125 and -0.75 as $f(-0.75)f(-0.8125) < 0$

$$x_5 = \frac{-0.75-0.8125}{2} = -0.78125$$

Example :

Carry out the first five iterations $f(x) = x \cos x - 2x^2 + 3x - 1$, $1.2 \leq x \leq 1.3$

Note: All the calculations should be done in the radians.

Solution:

$$f(x) = x \cos x - 2x^2 + 3x - 1$$

$$f(1.2) = 1.2 \cos 1.2 - 2(1.2)^2 + 3(1.2) - 1$$

$$= 1.2(0.3623) - 2(1.44) + 3.6 - 1 = 0.1548 > 0$$

$$f(1.3) = 1.3 \cos 1.3 - 2(1.3)^2 + 3(1.3) - 1$$

$$= 1.3(0.2674) - 2(1.69) + 3.9 - 1 = -0.1322 < 0$$

as $f(1.2)f(1.3) < 0$ so the root lies between both

$$x_2 = \frac{1.2+1.3}{2} = 1.25$$

$$f(1.25) = 1.25 \cos 1.25 - 2(1.25)^2 + 3(1.25) - 1$$

$$= 1.25(0.3153) - 2(1.5625) + 3.75 - 1 = 0.0191 > 0$$

as $f(1.25)f(1.3) < 0$ so the root lies between both

$$x_3 = \frac{1.25+1.3}{2} = 1.275$$

$$f(1.275) = 1.275 \cos 1.275 - 2(1.275)^2 + 3(1.275) - 1$$

$$= 1.275(0.2915) - 2(1.6256) + 3.825 - 1 = -0.0545 < 0$$

as $f(1.25)f(1.275) < 0$ so the root lies between both

$$x_4 = \frac{1.25+1.275}{2} = 1.2625$$

$$f(1.2625) = 1.2625 \cos 1.2625 - 2(1.2625)^2 + 3(1.2625) - 1$$

$$= 1.2625(0.3034) - 2(1.5939) + 3.7875 - 1 = -0.0172 < 0$$

as $f(1.25)f(1.2625) < 0$ so the root lies between both

$$x_5 = \frac{1.25+1.2625}{2} = 1.25625$$

$$f(1.25625) = 1.25625 \cos 1.25625 - 2(1.25625)^2 + 3(1.25625) - 1$$

$$= 1.25625(0.3093) - 2(1.5781) + 3(1.25625) - 1 = 0.00108 > 0$$

as $f(1.25625)f(1.2625) < 0$ so the root lies between both

$$x_6 = \frac{1.25625+1.2625}{2} = 1.259375$$

Regula-Falsi method (Method of false position)

Here we choose two points x_n and x_{n-1} such that $f(x_n)$ and $f(x_{n-1})$ have opposite signs. Intermediate value property suggests that the graph of the $y=f(x)$ crosses the x -axis between these two points and therefore, a root lies between these two points. Thus to find the real root of $f(x)=0$ using Regula-Falsi method, we replace the part of the curve between the points $A(x_n, f(x_n))$ and $B(x_{n-1}, f(x_{n-1}))$ by a chord in the interval and we take the point of intersection of this chord with x -axis as initial approximation. Now, the equation of the chord joining the points A and B is,

$$\frac{y - f(x_n)}{f(x_{n-1}) - f(x_n)} = \frac{x - x_n}{x_{n-1} - x_n}$$

Setting $y=0$ in the above equation we get

$$x = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

Hence the first approximation to the root is given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

We observe that $f(x_{n-1})$ and $f(x_{n+1})$ are of opposite signs thus it is possible to apply the above procedure, to determine the line through B and A_1 and so on.

Hence for successive approximation to the root above formula is used.

Example

Use the Regula-Falsi method to compute a real root of the equation $x^3 - 9x + 1 = 0$,

- (i) if the root lies between 2 and 4
- (ii) if the root lies between 2 and 3.

Comment on the results.

Solution

Let

$$f(x) = x^3 - 9x + 1$$

$$f(2) = 2^3 - 9(2) + 1 = 8 - 18 + 1 = -9 \text{ and } f(4) = 4^3 - 9(4) + 1 = 64 - 36 + 1 = 29.$$

Since $f(2)$ and $f(4)$ are of opposite signs, the root of $f(x) = 0$ lies between 2 and 4.

Taking $x_1 = 2$, $x_2 = 4$ and using Regula-Falsi method, the first approximation is given by

$$\begin{aligned} x_3 &= x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 4 - \frac{4 - 2}{29 - (-9)} (29) = 4 - \frac{2(29)}{38} \\ &= 4 - \frac{58}{38} = 4 - 1.5263 = 2.4736 \end{aligned}$$

Now

$$f(x_3) = 2.4736^3 - 9(2.4736) + 1 = 15.13520 - 22.2624 + 1 = -6.12644.$$

Since $f(x_2)$ and $f(x_3)$ are of opposite signs, the root lies between x_2 and x_3 .

The second approximation to the root is given as

$$\begin{aligned} x_4 &= x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 2.4736 - \frac{2.4736 - 2}{-6.12644 - 29} (-6.12644) \\ &= 2.4736 - \frac{-1.5264}{-35.12644} (-6.12644) = 2.4736 - (0.04345)(-6.12644) \\ &= 2.4736 + 0.26619 = 2.73989 \end{aligned}$$

Therefore

$$f(x_4) = 2.73989^3 - 9(2.73989) + 1 = 20.5683 - 24.65901 + 1 = -3.090707.$$

Now, since $f(x_2)$ and $f(x_4)$ are of opposite signs, the third approximation is obtained from

$$\begin{aligned} x_5 &= x_4 - \frac{x_4 - x_2}{f(x_4) - f(x_2)} f(x_4) = 2.73989 - \frac{2.73989 - 2}{-3.090707 - 29} (-3.090707) = 2.86125 \\ &= 2.73989 - \frac{-1.26011}{-32.090707} (-3.090707) = 2.73989 + 0.039267(3.090707) = 2.73989 + 0.121363 = 2.86125 \end{aligned}$$

Now

$$f(x_5) = 2.86125^3 - 9(2.86125) + 1 = 23.42434 - 25.75125 + 1 = -1.326868.$$

(ii)

Here

$$f(x) = x^3 - 9x + 1$$

$$f(2) = 2^3 - 9(2) + 1 = 8 - 18 + 1 = -9 \text{ and } f(3) = 3^3 - 9(3) + 1 = 27 - 27 + 1 = 1.$$

Since $f(2)$ and $f(3)$ are of opposite signs, the root of $f(x) = 0$ lies between 2 and 3.

Taking $x_1 = 2$, $x_2 = 3$ and using Regula-Falsi method, the first approximation is given by

$$\begin{aligned} x_3 &= x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 3 - \frac{3 - 2}{1 - (-9)} (-9) \\ &= 3 - \frac{1}{10} = 2.9 \end{aligned}$$

$$f(x_3) = 2.9^3 - 9(2.9) + 1 = 24.389 - 26.1 + 1 = -0.711$$

Since $f(x_2)$ and $f(x_3)$ are of opposite signs, the root lies between x_2 and x_3 .

The second approximation to the root is given as

$$\begin{aligned} x_4 &= 2.9 - \frac{2.9 - 3}{-0.711 - 1} (-0.711) = 2.9 - \frac{-0.1}{-1.711} (-0.711) = 2.9 - \frac{-0.1}{-1.711} (-0.711) \\ &= 2.9 - (0.05844)(-0.711) = 2.9 + 0.04156 = 2.94156 \end{aligned}$$

$$f(x_4) = -0.0207$$

$$f(x_4) = 2.94156^3 - 9(2.94156) + 1 = 25.45265 - 26.47404 + 1 = -0.0207$$

Now, we observe that $f(x_2)$ and $f(x_4)$ are of opposite signs; the third approximation is obtained from

$$x = 2.94156 - \frac{2.94156 - 3}{-0.0207 - 1}(-0.0207) = 2.94156 - \frac{-0.05844}{-1.0207}(-0.0207)$$

$$= 2.94156 - (-0.05725)(-0.0207) = 2.94275$$

$$f(x_5) = 2.94275^3 - 9(2.94275) + 1 = 25.48356 - 26.48475 + 1 = -0.0011896$$

We observe that the value of the root as a third approximation is evidently different in both the cases, while the value of x_5 , when the interval considered is $(2, 3)$, is closer to the root.

Important observation: The initial interval (x_1, x_2) in which the root of the equation lies should be sufficiently small.

Example

Use Regula-Falsi method to find a real root of the equation $\ln x - \cos x = 0$ accurate to four decimal places after three successive approximations.

Note: here is a transcendental equation all the calculation should be done in the radians mode

Sol:

$$f(x) = \ln x - \cos x$$

we have

$$f(1) = \ln 1 - \cos(1) = 0 - 0.540302 = -0.540302 < 0$$

$$f(2) = \ln 2 - \cos(2) = 0.69315 - 0.41615 = 1.109$$

As $f(1)f(2) < 0$ so the root lies between 1 and 2

the first approximation is obtained from

$$x_3 = 2 - \frac{2 - 1}{1.109 + 0.540302}(1.109)$$

$$= 2 - \frac{1.1093}{1.6496} = 1.3275$$

$$f(x_3) = \ln 1.3275 - \cos 1.3275 = 0.2833 - 0.2409 = 0.0424$$

Now, since $f(x_1)$ and $f(x_3)$ are of opposite signs, the second approximation is obtained as

$$x_4 = 1.3275 - \frac{(1.3275)(0.0424)}{0.0424 + 0.5403}$$

$$= 1.3037$$

$$f(x_4) = \ln 1.3037 - \cos 1.3037 = 1.24816 \times 10^{-3}$$

Similarly, we observe that $f(x_1)$ and $f(x_4)$ are of opposite signs, so, the third approximation is given by

$$x_5 = 1.3037 - \frac{(0.3037)(0.001248)}{0.001248 + 0.5403}$$

$$= 1.3030$$

$$f(x_5) = 0.6245 \times 10^{-4}$$

The required real root is 1.3030

Example:

Use method of false position to solve $e^{-x} + 2^{-x} + 2 \cos x - 6 = 0 \quad 1 \leq x \leq 2$

Solution:

$$f(x) = e^{-x} + 2^{-x} + 2 \cos x - 6$$

$$x_0 = 1, \quad x_1 = 2$$

now

$$x_{n+1} = \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$f(1) = e^{-1} + 2^{-1} + 2 \cos 1 - 6 = 2.7182 + 0.5 + 2(0.5403) - 6 = -1.7011$$

$$f(2) = e^{-2} + 2^{-2} + 2 \cos 2 - 6 = 7.3886 + 0.25 + 2(-0.4161) - 6 = 0.8068$$

now for $n=1$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) = 2 - \frac{2-1}{0.8068+1.7011} (0.8068)$$

$$x_2 = 2 - \frac{1}{2.5079} (0.8068) = 1.6783$$

$$f(1.6783) = e^{-1.6783} + 2^{-1.6783} + 2 \cos(1.6783) - 6 = -0.5457$$

now for $n=2$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 1.6783 - \frac{1.6783 - 2}{(-0.5457) - 0.8068} (-0.5457)$$

$$x_3 = 1.6783 - \frac{(-0.3217)}{(-1.3525)} (-0.5457) = 1.6783 + 0.12979 = 1.8081$$

$$f(1.8081) = e^{1.6783} + 2^{-1.8081} + 2 \cos(1.8081) - 6 = -0.8575$$

now for $n = 3$

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 1.8081 - \frac{1.8081 - 1.6783}{(-0.08575) + 0.5457} (-0.08575)$$

$$x_3 = 1.8081 - \frac{0.1298}{0.45995} (-0.08575) = 1.6783 + 0.12979 = 1.8323$$

$$f(1.8323) = e^{1.8323} + 2^{-1.8323} + 2 \cos(1.8323) - 6 = 0.1199$$

now for $n = 4$

$$x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4) = 1.8323 - \frac{1.8323 - 1.8081}{0.01199 + 0.08575} (0.01199)$$

$$x_5 = 1.8323 - \frac{0.0242}{0.09774} (0.01199) = 1.8323 - 0.00296 = 1.8293$$

$$f(1.8293) = e^{1.8293} + 2^{-1.8293} + 2 \cos(1.8293) - 6 = -0.000343$$

now for $n = 5$

$$x_6 = x_5 - \frac{x_5 - x_4}{f(x_5) - f(x_4)} f(x_5) = 1.8293 - \frac{1.8293 - 1.8323}{-0.000343 - 0.01199} (-0.000343)$$

$$x_6 = 1.8293 - \frac{(-0.003)}{-0.012333} (-0.000343) = 1.8293$$

Example:

Solve the equation $2x \cos 2x - (x - 2)^2 = 0$ $2 \leq x \leq 3$ Perform only three iterations.

Solution

$$f(x) = 2x \cos 2x - (x - 2)^2$$

here $x_0 = 2$ and $x_1 = 3$

$$\text{so } f(2) = 2(2) \cos 4 - (2 - 2)^2 = 4 \cos 4 = -2.6146$$

$$f(3) = 2(3) \cos 2(3) - (3 - 2)^2 = 6 \cos 6 - 1 = 4.7610$$

$$\text{here } x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

for $n = 1$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) = 3 - \frac{3 - 2}{4.7610 - 2.4146} (4.7610)$$

$$= 3 - \frac{1}{7.3756}(4.7610) = 3 - 0.6455 = 2.3545$$

$$f(2.3545) = 2(2.3545) \cos 2(2.3545) - (2.3545 - 2)^2 = 4.709 \cos 4.709 - 0.1257 = -0.1416$$

for $n = 2$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 2.3545 - \frac{2.3545 - 3}{-0.1416 - 4.7610} (-0.1416) \\ = 2.3731$$

$$f(2.3713) = 2(2.3713) \cos 2(2.3713) - (2.3713 - 2)^2 = 4.7462 \cos 4.7462 - 0.1392 = -0.1392$$

for $n = 3$

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 2.3713 - \frac{2.3731 - 2.3545}{0.0212 + 0.1416} (0.0212) \\ = 2.3707$$

$$f(2.3707) = 2(2.3707) \cos 2(2.3707) - (2.3707 - 2)^2 = 4.7414 \cos 4.7412 - 0.1374 = 0.00013$$

for $n = 4$

$$x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4) = 2.3707 - \frac{2.3707 - 2.3731}{0.00013 - 0.0212} (0.00013) \\ = 2.3707$$

Example

Using Regula-Falsi method, find the real root of the following equation correct, to three decimal places: $x \log_{10} x = 1.2$

Solution:

$$\text{Let } f(x) = x \log_{10} x - 1.2$$

$$f(2) = 2 \log_{10} 2 - 1.2 = -0.5979,$$

$$f(3) = 3 \log_{10} 3 - 1.2 = 0.2314.$$

Since $f(2)$ and $f(3)$ are of opposite signs, the real root lies between $x_1 = 2$, $x_2 = 3$.

The first approximation is obtained from

$$\begin{aligned} x_3 &= x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 3 - \frac{3 - 2}{0.2314 + 0.5979} (0.2314) \\ &= 3 - \frac{0.2314}{0.8293} = 2.72097 \end{aligned}$$

$$f(x_3) = \text{Let } f(x) = 2.72097 \log_{10} 2.72097 - 1.2 = -0.01713.$$

Since $f(x_2)$ and $f(x_3)$ are of opposite signs, the root of $f(x) = 0$ lies between x_2 and x_3 . Now, the second approximation is given by

$$\begin{aligned} x_4 &= x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 2.72097 - \frac{2.72097 - 3}{-0.01713 - 0.2314} (-0.01713) = 2.7402 \\ f(x_4) &= 2.7402 \log_{10} 2.7402 - 1.2 = 3.8905 \times 10^{-4} \end{aligned}$$

Thus, the root of the given equation correct to three decimal places is 2.740

NOTE: Here if TOL is given then we can simply find the value of TOL by subtracting both the consecutive roots and write it in the exponential notation if the required TOL is obtained then we stop.

Method of Iteration

Method of iterations can be applied to find a real root of the equation $f(x) = 0$ by rewriting the same in the form.

$$x = \phi(x)$$

Let $x = x_0$ be the initial approximation to the actual root, say, α of the equation. Then the first approximation is $x_1 = \phi(x_0)$ and the successive approximations are $x_2 = \phi(x_1)$

$x_3 = \phi(x_2)$, $x_4 = \phi(x_3)$, ..., $x_n = \phi(x_{n-1})$ if sequence of the approximate roots, $x_1, x_2, x_3, \dots, x_n$ converges to α it is taken as the root of the equation $f(x) = 0$.

For convergence purpose the initial approximation is to be done carefully. The choice of the x_0 is done according to the theorem.

Theorem

If α be a root of $f(x)=0$ which is equivalent to $x = \phi(x)$ I be any interval containing the point $x=\alpha$ and $|\phi'(x)| < 1 \forall x \in I$, then the sequence of approximations $x_1, x_2, x_3, \dots, x_n$ will converge to the root α provided that the initial approximation x_0 is chosen in I

Example,

$$f(x) = \cos x - 2x + 3 = 0.$$

It can be

$$\text{Rewritten as } x = \frac{1}{2}(\cos x + 3) = \phi(x)$$

$$\phi(x) = \frac{1}{2}(\cos x + 3)$$

$$f(x) = \cos x - 2x + 3 = 0.$$

$$f(1) = \cos 1 - 2(1) + 3 = 1.54030 > 0$$

$$f(2) = \cos 2 - 2(2) + 3 = -0.41614 < 0$$

so root lies between 1 and 2

$$\phi'(x) = -\frac{1}{2}(\sin x)$$

both $\phi'(1)$ and $\phi'(2) < 1$ so the method of iterations can be applied

let $x_0 = 1.5$

$$x_1 = \frac{1}{2}(\cos x_0 + 3) = \frac{1}{2}(\cos(1.5) + 3) = 1.999825$$

$$x_2 = \frac{1}{2}(\cos x_1 + 3) = \frac{1}{2}(\cos(1.999825) + 3) = 1.999695$$

$$x_3 = \frac{1}{2}(\cos x_2 + 3) = \frac{1}{2}(\cos(1.999695) + 3) = 1.999695$$

So this is the required root correct up to 5 places of decimal.

Example

Find the real root of the equation $x^3 + x^2 - 1 = 0$ by method of iterations

Solution

$$\text{let } f(x) = x^3 + x^2 - 1$$

$$\text{now } f(0) = 0^3 + 0^2 - 1 = -1 < 0$$

$$f(1) = 1^3 + 1^2 - 1 = 1 > 0$$

hence a real root lies between 0 and 1

here

$$x^3 + x^2 - 1 = 0$$

$$x^2(x+1) = 1$$

$$x^2 = \frac{1}{(x+1)} \Rightarrow x = \frac{1}{\sqrt{x+1}} = \phi(x)$$

$$\text{here } \phi'(x) = -1/2[1/(x+1)^{3/2}]$$

$$\phi'(0) = 1/2 < 1 \text{ and } \phi'(1) = 1/2^2 < 1$$

so $\phi'(x) < 1$ for all the values in the interval

let $x_0 = 0.65$

$$x_1 = \phi(x_0) = \frac{1}{\sqrt{x_0+1}} = \frac{1}{\sqrt{1.65}} = 0.7784989$$

$$x_2 = \phi(x_1) = \frac{1}{\sqrt{x_1+1}} = \frac{1}{\sqrt{1.7784989}} = 0.7498479$$

$$x_3 = \phi(x_2) = \frac{1}{\sqrt{x_2+1}} = \frac{1}{\sqrt{1.7498479}} = 0.7559617$$

$$x_4 = \phi(x_3) = \frac{1}{\sqrt{x_3+1}} = \frac{1}{\sqrt{1.7559617}} = 0.7546446$$

$$x_5 = \phi(x_4) = \frac{1}{\sqrt{x_4+1}} = \frac{1}{\sqrt{1.7546446}} = 0.7549278$$

$$x_6 = \phi(x_5) = \frac{1}{\sqrt{x_5+1}} = \frac{1}{\sqrt{1.7549278}} = 0.7548668$$

$$x_7 = \phi(x_6) = \frac{1}{\sqrt{x_6+1}} = \frac{1}{\sqrt{1.7548668}} = 0.7548799$$

$$x_8 = \phi(x_7) = \frac{1}{\sqrt{x_7+1}} = \frac{1}{\sqrt{1.7548799}} = 0.7548771$$

$$x_9 = \phi(x_8) = \frac{1}{\sqrt{x_8+1}} = \frac{1}{\sqrt{1.7548771}} = 0.7548777$$

$$x_{10} = \phi(x_9) = \frac{1}{\sqrt{x_9+1}} = \frac{1}{\sqrt{1.7548777}} = 0.7548776$$

$$x_{11} = \phi(x_{10}) = \frac{1}{\sqrt{x_{10}+1}} = \frac{1}{\sqrt{1.7548776}} = 0.7548776$$

hence root is 0.7548776

Note: In this question the accuracy up to 7 places is acquired or here the TOL is 10^{-7}

Example

Find a real root of the equation $\cos x = 3x - 1$ correct to seven places of decimal.

Solution

Here it is a transcendental function and all the calculation must be done in the radians mode and value of pi should be 3.14

$$f(x) = \cos x - 3x + 1$$

$$f(0) = \cos 0 - 3(0) + 1 = 1 > 0$$

$$f(\pi/2) = \cos(1.57) - 3(1.57) + 1 = 0.0007963 - 4.71 + 1 = -3.7092037 < 0$$

so a real root lies between 0 and $\pi/2$

$$\text{here } \phi(x) = \frac{1}{3}(\cos x + 1)$$

$$\text{we have } \phi'(x) = -\frac{1}{3}\sin x$$

it is clearly less than 1 as sin is a bounded function and its values lie between -1 and 1

hence iteration method can be applied

let $x_0 = 0.5$ be the initial approximation then

$$x_1 = \phi(x_0) = \frac{1}{3}[\cos(0.5) + 1] = 0.6258608$$

$$x_2 = \phi(x_1) = \frac{1}{3}[\cos(0.6258608) + 1] = 0.6034863$$

$$x_3 = \phi(x_2) = \frac{1}{3}[\cos(0.6034863) + 1] = 0.6077873$$

$$x_4 = \phi(x_3) = \frac{1}{3}[\cos(0.6077873) + 1] = 0.6069711$$

$$x_5 = \phi(x_4) = \frac{1}{3}[\cos(0.6069711) + 1] = 0.6071264$$

$$x_6 = \phi(x_5) = \frac{1}{3}[\cos(0.6071264) + 1] = 0.6070969$$

$$x_7 = \phi(x_6) = \frac{1}{3} [\cos(0.6070969) + 1] = 0.6071025$$

$$x_8 = \phi(x_7) = \frac{1}{3} [\cos(0.6071025) + 1] = 0.6071014$$

$$x_9 = \phi(x_8) = \frac{1}{3} [\cos(0.6071014) + 1] = 0.6071016$$

$$x_{10} = \phi(x_9) = \frac{1}{3} [\cos(0.6071016) + 1] = 0.6071016$$

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Newton -Raphson Method

This method is one of the most powerful method and well known methods, used for finding a root of $f(x)=0$ the formula many be derived in many ways the simplest way to derive this formula is by using the first two terms in Taylor's series expansion of the form,

$$f(x_{n+1}) = f(x_n) + (x_{n+1} - x_n) f'(x_n)$$

setting $f(x_{n+1}) = 0$ gives,

$$f(x_n) + (x_{n+1} - x_n) f'(x_n) = 0$$

thus on simplification, we get ,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \text{ for } n = 0, 1, 2, \dots$$

Geometrical interpretation

Let the curve $f(x)=0$ meet the x-axis at $x=\alpha$.it means that α is the original root of the $f(x)=0$. Let x_0 be the point near the root α of the equation $f(x)=0$ then the equation of the tangent $P_0[x_0, f(x_0)]$ is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

This cuts the x-axis at $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

This is the first approximation to the root α .if $P_1[x_1, f(x_1)]$ is the point corresponding to x_1 on the curve then the tangent at P_1 is

$$y - f(x_1) = f'(x_1)(x - x_1)$$

This cuts the x-axis at $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

This is the second approximation to the root α .Repeating this process we will get the root α with better approximations quite rapidly.

Note:

1. When $f'(x)$ very large .i.e. is when the slope is large, then h will be small (as assumed) and hence, the root can be calculated in even less time.
2. If we choose the initial approximation x_0 close to the root then we get the root of the equation very quickly.
3. The process will evidently fail if $f'(x) = 0$ is in the neighborhood of the root. In such cases the regula falsi method should be used.

4. If the initial approximation to the root is not given, choose two say, a and b, such that $f(a)$ and $f(b)$ are of opposite signs. If $|f(a)| < |f(b)|$ then take a as the initial approximation.
5. Newton's Raphson method is also referred to as the method of tangents.

Example

Find a real root of the equation $x^3 - x - 1 = 0$ using Newton - Raphson method, correct to four decimal places.

Solution

$$f(x) = x^3 - x - 1$$

$$f(1) = 1^3 - 1 - 1 = -1 < 0$$

$$f(2) = 2^3 - 2 - 1 = 8 - 2 - 1 = 5 > 0$$

so the root lies between 1 and 2

$$\text{here } f'(x) = 3x^2 - 1 \quad \text{and} \quad f''(x) = 6x$$

$$f'(1) = 3 \cdot 1^2 - 1 = 2$$

$$f'(2) = 3 \cdot 2^2 - 1 = 11$$

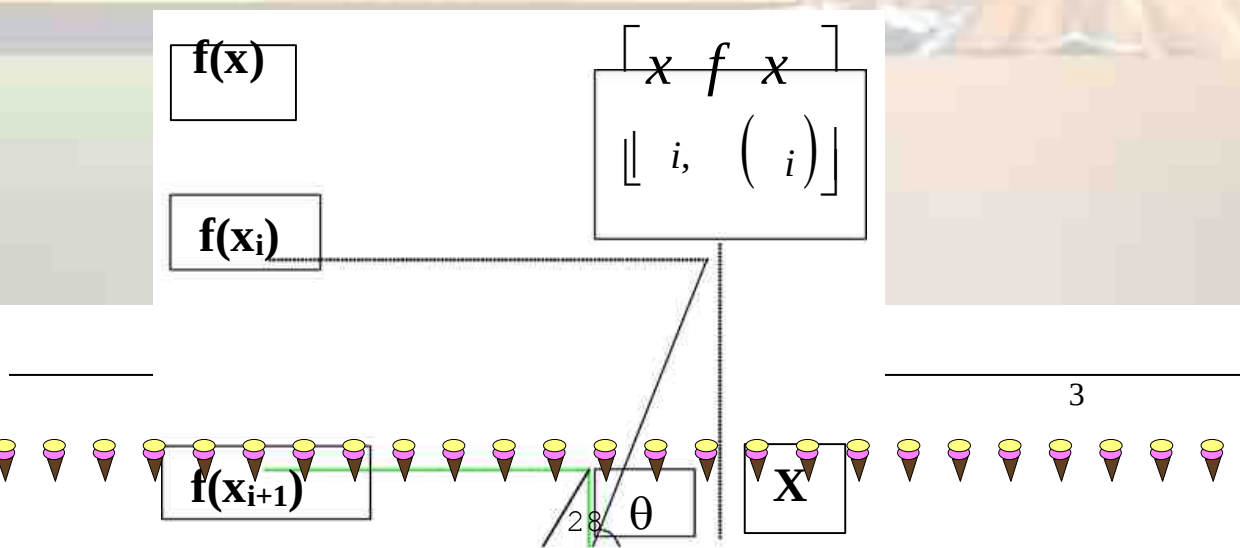
here

$$f''(1) = 6$$

$$f''(2) = 6(2) = 12$$

here $f(2)$ and $f''(2)$ have the same signs so $x_0 = 2$

The second approximation is computed using Newton-Raphson method as



Draw backs of N-R Method

Divergence at inflection points:

If the selection of a guess or an iterated value turns out to be close to the inflection point of $f(x)$,
[where $f''(x) = 0$],
the roots start to diverge away from the root.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Division of zero or near zero:

If an iteration, we come across the division by zero or a near-zero number, then we get a large magnitude for the next value, x_{i+1} .

Root jumping:

In some case where the function $f(x)$ is oscillating and has a number of roots, one may choose an initial guess close to a root. However, the guesses may jump and converge to some other root.

Oscillations near local maximum and minimum:

Results obtained from N-R method may oscillate about the local max or min without converging on a root but converging on the local max or min. Eventually, it may lead to division to a number close to zero and may diverge.

Convergence of N-R Method

Let us compare the N-R formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

with the general iteration formula

$$x_{n+1} = \phi(x_n),$$

$$\phi(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\phi(x) = x - \frac{f(x)}{f'(x)}$$

The iteration method converges if

$$|\phi'(x)| < 1.$$

Therefore, N-R formula converges, provided

$$|f(x)f''(x)| < [f'(x)]^2$$

in the interval considered.

Newton-Raphson formula therefore converges, provided the initial approximation x_0 is chosen sufficiently close to the root and are continuous and bounded in any small interval containing the root.

Definition

Let $x_n = \alpha + \varepsilon_n$, $x_{n+1} = \alpha + \varepsilon_{n+1}$

where α is a root of $f(x) = 0$.

If we can prove that $\varepsilon_{n+1} = K\varepsilon_n^p$,

where K is a constant and ε_n is the error involved at the n -th step, while finding the root by an iterative method, then the rate of convergence of the method is p .

The N-R method converges quadratically

$$x_n = \alpha + \varepsilon_n,$$

$$x_{n+1} = \alpha + \varepsilon_{n+1}$$

where α is a root of $f(x) = 0$ and ε_n is the error involved at the n -th step, while finding the root by N-R formula

$$\alpha + \varepsilon_{n+1} = \alpha + \varepsilon_n - \frac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)}$$

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)} = \frac{\varepsilon_n f'(\alpha + \varepsilon_n) - f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)}$$

Using Taylor's expansion, we get

$$x_{n+1} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{1}{2} \left(x_n + \frac{N}{x_n} \right)$$

Example

Evaluate $\sqrt{12}$, by Newton's formula.

Solution

Since

$$\sqrt{9} = 3, \sqrt{16} = 4,$$

We take

$$x_0 = (3+4)/2 = 3.5$$

$$x_1 = \frac{1}{2} \left(x_0 + \frac{N}{x_0} \right) = \frac{1}{2} \left(3.5 + \frac{12}{3.5} \right) = 3.4643$$

$$x_2 = \frac{1}{2} \left(3.4643 + \frac{12}{3.4643} \right) = 3.4641$$

$$x_3 = \frac{1}{2} \left(3.4641 + \frac{12}{3.4641} \right) = 3.4641$$

Hence

$$\sqrt{12} = 3.4641.$$

Here in this solution we use the iterative scheme developed in the above example and simply replace the previous value in the next iteration and finally come to the result.

Example

Find the first three iteration of the equation $f(x) = x - 0.8 - 0.2 \sin x$ in the interval $[0, \pi/2]$.

Solution

$$f(0) = 0 - 0.8 - 0.2 \sin(0) = 0 - 0.8 - 0.2(0) = -0.8$$

$$\begin{aligned} f(1.57) &= 1.57 - 0.8 - 0.2 \sin(1.75) \\ &= 1.57 - 0.8 - 0.2(0.99999) \\ &= 1.57 - 0.8 - 0.199998 = 0.570002 \end{aligned}$$

$$f'(x) = 1 - 0.2 \cos x$$

$$f'(0) = 1 - 0.2 \cos(0) = 1 - 0.2 = 0.8$$

here $|f(0)|$ is greater than $x_0 = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{-0.8}{0.8} = 1$$

now

$$\begin{aligned} f(1) &= 1 - 0.8 - 0.2 \sin(1) \\ &= 1 - 0.8 - 0.1683 \\ &= 0.0317 \end{aligned}$$

$$f'(x) = 1 - 0.2 \cos x$$

$$f'(1) = 1 - 0.2 \cos(1) = 1 - 0.1081 = 0.8919$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{0.0317}{0.8919} = 1 - 0.0355 = 0.9645$$

$$\begin{aligned} f(0.9645) &= 0.9645 - 0.8 - 0.2 \sin(0.9645) \\ &= 0.9645 - 0.8 - 0.1645 \\ &= 0.0002 \end{aligned}$$

$$f'(x) = 1 - 0.2 \cos x$$

$$f'(0.9645) = 1 - 0.2 \cos(0.9645) = 1 - 0.11396 = 0.88604$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.9645 - \frac{0.0002}{0.88604} = 0.9645 - 0.00022 = 0.9643$$

NOTE: In the above question the function is a transcendental function and we have to perform all the calculation in the radians mode and the value of pi should be taken as 3.14

Example

Consider the equation $f(x) = 4x \cos x - (x - 2)^2$ find the root of the equation in the range $0 \leq x \leq 8$

Solution

$$f(x) = 4x \cos x - (x - 2)^2$$

here

$$f(0) = 4(0) \cos 2(0) - (0 - 2)^2 = -4$$

$$\begin{aligned} f(8) &= 4(8) \cos 2(8) - (8 - 2)^2 \\ &= 32 \cos 16 - (6)^2 = -30.6451 - 36 = -66.6451 \end{aligned}$$

$$\begin{aligned} f'(x) &= 4 \cos 2x - 8x \sin 2x - 2(x - 2) \\ &= 4 \cos 2x - 8x \sin 2x - 2(x - 2) \end{aligned}$$

$$\begin{aligned} f'(8) &= 4 \cos 16 - 64 \sin 16 - 2(8 - 2) \\ &= -3.8306 + 18.4250 - 12 = 2.5952 \end{aligned}$$

since $|f(8)|$ is greater so $x_0 = 8$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 8 - \frac{(-66.6451)}{2.5952} = 33.6801$$

$$\begin{aligned} f(33.6801) &= 4(33.6801) \cos 2(33.6801) - (33.6801 - 2)^2 \\ &= -24.6545 - 1003.6 = -1028.25 \end{aligned}$$

$$\begin{aligned} f'(33.6801) &= 4 \cos 2(33.6801) - 8(33.6801) \sin 2(33.6801) - 2(33.6801 - 2) \\ &= -0.7320 + 264.89 = -63.3602 \end{aligned}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 33.6801 + \frac{1028.25}{200.79} = 38.8011$$

$$\begin{aligned} f(38.8011) &= 4(38.8011) \cos 2(38.8011) - (38.8011 - 2)^2 \\ &= -91.8361 - 1354.3 = 1446.14 \end{aligned}$$

$$\begin{aligned} f'(38.8011) &= 4 \cos 2(38.8011) - 8(38.8011) \sin 2(38.8011) - 2(38.8011 - 2) \\ &= -2.3668 - 250.236 - 73.6022 = -326.205 \end{aligned}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 38.8011 + \frac{1446.14}{-326.205} = 38.8011 + 4.4332 = 43.2343$$

Example

Perform three iteration of the equation $\ln(x-1) + \cos(x-1) = 0$ when $1.2 \leq x \leq 2$. Use Newton Raphson method to calculate the root.

Solution

Here

$$\ln(x-1) + \cos(x-1) = 0 \text{ when } 1.2 \leq x \leq 2$$

$$f(x) = \ln(x-1) + \cos(x-1)$$

$$f(x) = \ln(x-1) + \cos(x-1)$$

$$f(1.2) = \ln(1.2-1) + \cos(1.2-1)$$

$$= -1.6094 + 0.9801 = -0.6293$$

$$f(2) = \ln(2-1) + \cos(2-1)$$

$$= 0 + 0.5403 = 0.5403$$

$$\text{now } f(x) = \ln(x-1) + \cos(x-1)$$

$$f'(x) = \frac{1}{x-1} - \sin(x-1)$$

$$f'(1.2) = \frac{1}{1.2-1} - \sin(1.2-1)$$

$$= 5 - 0.1986 = 4.8014$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.2 - \frac{-0.6293}{4.8014} = 1.2 + 0.1311 = 1.3311$$

$$f(1.3311) = \ln(1.3311-1) + \cos(1.3311-1)$$

$$= -1.1053 + 0.9457 = -0.1596$$

$$f'(1.3311) = \frac{1}{1.3311-1} - \sin(1.3311-1)$$

$$= 3.0202 - 0.3251 = 2.6951$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.3311 - \frac{-0.1596}{2.6951} = 1.3311 + 0.0592 = 1.3903$$

$$f(1.3903) = \ln(1.3903-1) + \cos(1.3903-1)$$

$$= -0.9408 + 0.9248 = -0.016$$

$$f'(1.3903) = \frac{1}{1.3903-1} - \sin(1.3903-1)$$

$$= 2.5621 - 0.3805 = 2.1816$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.3903 - \frac{-0.016}{2.1816} = 1.3903 + 0.0073 = 1.3976$$

Secant Method

The secant method is modified form of Newton-Raphson method. If in Newton-Raphson method; we replace the derivative $f'(x_n)$ by the difference ratio, i.e.,

$$f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Where x_n and x_{n-1} are two approximations of the root we get

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \\ &= \frac{x_n f(x_n) - x_n f(x_{n-1}) - f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \\ &= \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \end{aligned}$$

Provided $f(x_n) \neq f(x_{n-1})$

This method requires two starting values x_0 and x_1 ; values at both these points are calculated which gives two points on the curve the next point on the curve are obtained by using the derived formula. We continue this procedure until we get the root of required accuracy.

Geometrical Interpretation

Geometrically, the secant method corresponds to drawing secants rather than tangents to obtain various approximations to root α ;

To obtain x_2 we find the intersection between the secant through the points $(x_0, f(x_0))$

And $(x_1, f(x_1))$ and the x-axis.

It is experienced that the secant method converges rather quickly. There is a possibility of the divergence if the two roots lie on the same side of the curve. The order of the

convergence of the secant method equal to $\frac{(1+\sqrt{5})}{2} = 1.618$, which shows that this

method has the order of convergence slightly inferior than that of Newton-Raphson method. In this method $f(x)$ is not required to change signs between the estimate.

Example

Do three iterations of secant method to find the root of

$$f(x) = x^3 - 3x + 1 = 0,$$

Taking

$$x_0 = 1, x_1 = 0.5$$

$$n = 1,$$

$$f(x_0) = f(1) = 1^3 - 3(1) + 1 = -1$$

$$f(x_1) = f(0.5) = 0.5^3 - 3(0.5) + 1 = 0.125 - 1.5 + 1 = -0.375$$

$$\begin{aligned} x_2 &= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \\ &= \frac{(1)(-0.375) - (0.5)(-1)}{-0.375 - (-1)} = 0.2 \end{aligned}$$

$$n = 2,$$

$$f(x_2) = f(0.2) = 0.2^3 - 3(0.2) + 1 = 0.008 - 0.6 + 1 = 0.408$$

$$\begin{aligned} x_3 &= \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \\ &= \frac{(0.5)(0.408) - 0.2(-0.375)}{0.408 - (-0.375)} = 0.3563 \end{aligned}$$

$$n = 3,$$

$$f(x_3) = f(0.3563) = 0.3563^3 - 3(0.3563) + 1 = 0.04523 - 1.0689 + 1 = -0.02367$$

$$\begin{aligned} x_4 &= \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} \\ &= \frac{(0.2)f(0.3563) - 0.3563f(0.2)}{f(0.3563) - f(0.2)} \end{aligned}$$

$$x_5 = 0.3473, f(x_5) = -0.0000096$$

$$\text{and } |x_5 - x_4| = 0.0004.$$

Though x_5 is not the true root, yet this is good approximation to the root and convergence is faster than bisection.

Example

Find the root of $2 \cosh x \sin x = 1$ using the secant method, with accuracy of 4 decimal point. Take 0.4 and 0.5 as the two starting values

$$f(x) = 2 \cosh x \sin x - 1$$

$$\text{now } f(x_0) = 2 \cosh x_0 \sin x_0 - 1$$

$$= 2 \cosh 0.4 \sin 0.4 - 1$$

$$= 2 \times 1.081 \times 0.3894 - 1 = -0.1580$$

$$f(x_1) = 2 \cosh x_1 \sin x_1 - 1$$

$$= 2 \cosh 0.5 \sin 0.5 - 1$$

$$= 2 \times 1.1276 \times 0.4794 - 1 = 0.0811$$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{0.4 \times 0.0811 - 0.5 \times -0.1580}{0.0811 + 0.1580} = \frac{0.03244 + 0.0790}{0.2391} = 0.4661$$

$$f(x_2) = 2 \cosh x_2 \sin x_2 - 1$$

$$= 2 \cosh 0.5 \sin 0.5 - 1$$

$$= 2 \times 1.1106 \times 0.4494 - 1 = -0.0811$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{0.5 \times -0.018 - 0.4661 \times 0.0811}{-0.0018 - 0.081} = \frac{0.009 - 0.0378}{-0.0828} = 0.4668$$

$$f(x_3) = 2 \cosh x_3 \sin x_3 - 1$$

$$= -0.00009$$

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$$= \frac{0.4661 \times -0.00009 - 0.4668 \times -0.0018}{-0.00009 + 0.0018} = \frac{-0.000042 + 0.00048}{-0.00171} = 0.4667$$

Comparison:

In secant method, we do not check whether the root lies in between two successive approximates x_{n-1} , and x_n .

This checking was imposed after each iteration, in Regula –Falsi method.

Muller's Method

In *Muller's method*, $f(x) = 0$ is approximated by a second degree polynomial; that is by a quadratic equation that fits through three points in the vicinity of a root. The roots of this quadratic equation are then approximated to the roots of the equation $f(x) = 0$. This method is iterative in nature and does not require the evaluation of derivatives as in Newton-

Raphson method. This method can also be used to determine both real and complex roots of $f(x) = 0$.

Suppose, x_{i-2}, x_{i-1}, x_i

be any three distinct approximations to a root

Of $f(x) = 0$.

$$f(x_{i-2}) = f_{i-2}, f(x_{i-1}) = f_{i-1}$$

$$f(x_i) = f_i.$$

Noting that any three distinct points in the (x, y) -plane uniquely; determine a polynomial of second degree.

A general polynomial of second degree is given by

Noting that any three distinct points in the (x, y) -plane uniquely; determine a polynomial of second degree.

A general polynomial of second degree is given by

$$f(x) = ax^2 + bx + c$$

Suppose, it passes through the points

$$(x_{i-2}, f_{i-2}), (x_{i-1}, f_{i-1}), (x_i, f_i)$$

Then the following equations will be satisfied

$$ax_{i-2}^2 + bx_{i-2} + c = f_{i-2}$$

$$ax_{i-1}^2 + bx_{i-1} + c = f_{i-1}$$

$$ax_i^2 + bx_i + c = f_i$$

Eliminating a, b, c , we obtain

$$\begin{vmatrix} x_{i-2}^2 & x_{i-2} & 1 & f_{i-2} \\ x_{i-1}^2 & x_{i-1} & 1 & f_{i-1} \\ x_i^2 & x_i & 1 & f_i \end{vmatrix} = 0$$

This can be written as

$$f = \frac{(x - x_{i-1})(x - x_i)}{(x_{i-2} - x_{i-1})(x_{i-2} - x_i)} f_{i-2} + \frac{(x - x_{i-2})(x - x_i)}{(x_{i-1} - x_{i-2})(x_{i-1} - x_i)} f_{i-1} + \frac{(x - x_{i-2})(x - x_{i-1})}{(x_i - x_{i-2})(x_i - x_{i-1})} f$$

We further define

$$\lambda = \frac{h}{h_i} = \frac{x - x_i}{x_i - x_{i-1}}$$

$$\lambda_i = \frac{h_i}{h_{i-1}}$$

$$\delta_i = 1 + \lambda_i$$

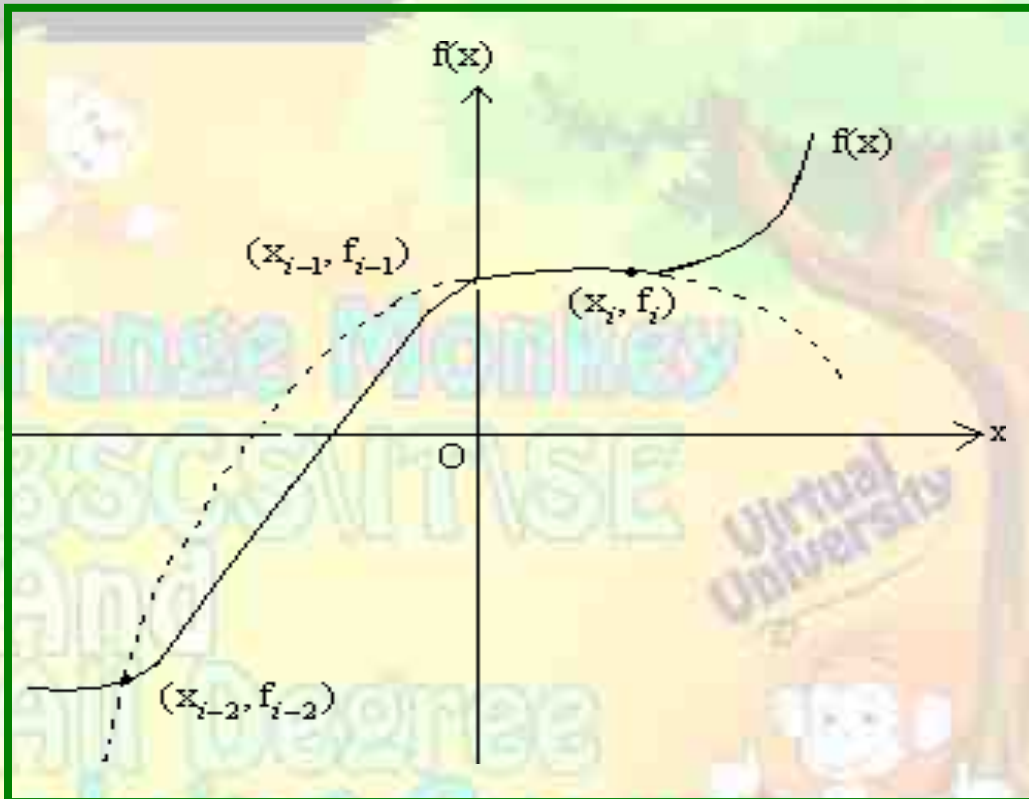


Fig: Quadratic polynomial

With these substitutions we get a simplified Equation as

$$f = \frac{1}{\delta_i} [\lambda(\lambda+1)\lambda^2 f_{i-2} - \lambda(\lambda+1+\lambda^{-1})\lambda \delta_i f_{i-1} + (\lambda+1+\lambda_i^{-1})\lambda_i f_i]$$

Or

$$f = \lambda^2 (f_{i-2} \lambda^2 - f_{i-1} \lambda \delta + f_i \lambda) \delta^{-1} + \lambda [f_{i-2} \lambda^2 - f_{i-1} \delta^2 + f_i (\lambda_i + \delta_i)] \delta_i^{-1} + f_i$$

To compute λ , set $f = 0$, we obtain
 $\lambda (f_{i-2} \lambda - f_{i-1} \delta + f_i) \lambda^2 + g_i \lambda + \delta_i f_i = 0$

Where
 $g_i = f_{i-2} \lambda^2 - f_{i-1} \delta^2 + f_i (\lambda_i + \delta_i)$

A direct solution will lead to loss of accuracy and therefore to obtain max accuracy we rewrite as:

$$\frac{f_i \delta_i}{\lambda^2} + \frac{g_i}{\lambda} + \lambda (f_{i-2} \lambda - f_{i-1} \delta + f_i) = 0$$

So that,

$$\frac{1}{\lambda} = \frac{-g_i \pm [g_i^2 - 4 f_i \delta_i \lambda (f_{i-2} \lambda - f_{i-1} \delta + f_i)]^{1/2}}{2 f_i \delta_i}$$

Or

$$\lambda = \frac{-2 f_i \delta_i}{g_i \pm [g_i^2 - 4 f_i \delta_i \lambda (f_{i-2} \lambda - f_{i-1} \delta + f_i)]^{1/2}}$$

Here, the positive sign must be so chosen that the denominator becomes largest in magnitude.

We can get a better approximation to the root, by using

$$x_{i+1} = x_i + h_i \lambda$$

Muller's Method

Muller's method is a generalization of the secant method, in the sense that it does not require the derivative of the function. It is an iterative method that requires three starting points $(p_0, f(p_0))$, $(p_1, f(p_1))$, and $(p_2, f(p_2))$. A parabola is constructed that passes through the three points; then the quadratic formula is used to find a root of the quadratic for the next approximation. It has been proved that near a simple root Muller's method converges faster than the secant method and almost as fast as Newton's method. The method can be used to find real or complex zeros of a function and can be programmed to use complex arithmetic.

Example

Do two iterations of Muller's method to solve $x^3 - 3x + 1 = 0$ starting with $x_0 = 0.5$, $x_1 = 0$, $x_2 = 1$

Solution

$$f(x_0) = f_0 = (0.5)^3 - 3(0.5) + 1 = -0.375$$

$$f(x_1) = f_1 = (0)^3 - 3(0) + 1 = 1$$

$$f(x_2) = f_2 = (1)^3 - 3(1) + 1 = -1$$

$$c = f_0 = -0.375$$

$$h_1 = x_1 - x_0 = 0.5$$

$$h_2 = x_2 - x_1 = 1$$

$$a = \frac{h_1 f_2 - (h_1 + h_2) f_1 + h_2 f_0}{h_1 h_2 (h_1 + h_2)}$$

$$= \frac{(0.5)(-1) - (-0.375) + (1)(-0.375)}{0.5(1)(1.5)} = 1.5$$

$$b = \frac{f_1 - f_0 - ah_1^2}{h_1} = -2$$

$$\therefore x = x_0 - \frac{2c}{b \pm \sqrt{b^2 - 4ac}}$$

$$= 0.5 - \frac{2(-0.375)}{-2 \pm \sqrt{4 + 4(1.5)(0.375)}}$$

$$= 0.5 - \frac{-0.75}{-2 \pm \sqrt{4 + 2.25}} = 0.33333 < 0.5$$

Take

$$x_2 = 0, x_0 = 0.33333, x_1 = 0.5$$

$$h_1 = x_1 - x_0 = 0.16667, h_2 = x_0 - x_2 = 0.33333$$

$$c = f_0 = f(0.33333)$$

$$= x_0^3 - 3x_0 + 1 = 0.037046$$

$$f_1 = x_1^3 - 3x_1 + 1 = -0.375$$

$$f_2 = x_2^3 - 3x_2 + 1 = 1$$

$$a = \frac{h_2 f_1 - (h_1 + h_2) f_0 + h_1 f_2}{h_1 h_2 (h_1 + h_2)} = \frac{0.023148}{0.027778}$$

$$= 0.8333$$

$$b = \frac{f_1 - f_0 - ah_1^2}{h_1} = -2.6$$

$$x = x_0 - \frac{2c}{b - \sqrt{b^2 - 4ac}}$$

$$= 0.33333 - \frac{0.074092}{-5.2236}$$

$$= 0.3475 > 0.33333 = x_0$$

For third iteration take,

$$x_2 = 0.33333, x_0 = 0.3475, x_1 = 0.5$$

Graeffe's Root Squaring Method

GRAEFFE'S ROOT SQUARING METHOD is particularly attractive for finding all the roots of a polynomial equation. Consider a polynomial of third degree $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$f(-x) = a_0 - a_1 x + a_2 x^2 - a_3 x^3$$

$$f(x)f(-x) = a_3^2 x^6 - (a_2^2 - 2a_1 a_3) x^4 + (a_1^2 - 2a_0 a_2) x^2 - a_0^2$$

$$f(x)f(-x) = a_3^2 t^3 - (a_2^2 - 2a_1 a_3) t^2 + (a_1^2 - 2a_0 a_2) t - a_0^2$$

The roots of this equation are squares or $2i$ ($i = 1$), powers of the original roots. Here $i = 1$ indicates that squaring is done once.

The same equation can again be squared and this squaring process is repeated as many times as required. After each squaring, the coefficients become large and overflow is possible as i increase.

Suppose, we have squared the given polynomial ' i ' times, then we can estimate the value of the roots by evaluating $2i$ root of

$$\left| \frac{a_i}{a_{i-1}} \right|, \quad i = 1, 2, \dots, n$$

Where n is the degree of the given polynomial.

The proper sign of each root can be determined by recalling the original equation. This method fails, when the roots of the given polynomial are repeated.

This method has a great advantage over other methods in that it does not require prior information about approximate values etc. of the roots. But it is applicable to polynomials only and it is capable of giving all the roots. Let us see the case of the polynomial equation having real and distinct roots. Consider the following polynomial equation

$$f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n \quad (1)$$

separating the even and odd powers of x and squaring we get

$$(x^n + a_2 x^{n-2} + a_4 x^{n-4} + \dots)^2 = (a_1 x^{n-1} + a_3 x^{n-3} + a_5 x^{n-5} + \dots)^2$$

putting $x^2 = y$ and simplifying we get the new equation

$$y^n + b_1 y^{n-1} + b_2 y^{n-2} + \dots + b_{n-1} y + b_n = 0 \quad (2)$$

$$b_1 = -a_1^2 + 2a_2$$

$$b_2 = a_1^2 - 2a_1 a_3 + 2a_2^2$$

$$\dots \dots \dots$$

$$b_n = (-1)^n a_n^2$$

if $p_1, p_2, \dots, p_n$ be the roots of eq 1 then the roots of the equation 2 are $p_1^2, p_2^2, \dots, p_n^2$

Example

Using Graeffe root squaring method, find all the roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$

Solution

Using Graeffe root squaring method, the first three squared polynomials are as under:

For $i = 1$, the polynomial is

$$x^3 - (36 - 22)x^2 + (121 - 72)x - 36$$

$$= x^3 - 14x^2 + 49x - 36$$

For $i = 2$, the polynomial is

$$x^3 - (196 - 98)x^2 + (2401 - 1008)x - 1296$$

$$= x^3 - 98x^2 + 1393x - 1296$$

For $i = 3$, the polynomial is

$$x^3 - (9604 - 2786)x^2 + (1940449 - 254016)x - 1679616$$

$$= x^3 - 6818x^2 + 16864333x - 1679616$$

The roots of polynomial are

$$\sqrt{\frac{36}{49}} = 0.85714, \quad \sqrt{\frac{49}{14}} = 1.8708, \quad \sqrt{\frac{14}{1}} = 3.7417$$

Similarly the roots of the polynomial 2 are

$$\sqrt[4]{\frac{1296}{1393}} = 0.9821, \quad \sqrt[4]{\frac{1393}{98}} = 1.9417, \quad \sqrt[4]{\frac{98}{1}} = 3.1464$$

Still better estimates of the roots obtained from polynomial (3) are

$$\sqrt[8]{\frac{1679616}{1686433}} = 0.99949, \quad \sqrt[8]{\frac{1686433}{6818}} = 1.99143, \quad \sqrt[8]{\frac{6818}{1}} = 3.0144$$

The exact values of the roots of the given polynomial are 1, 2 and 3.

Example

Apply Graeffe's root squaring method to solve the equation $x^3 - 8x^2 + 17x - 10 = 0$

Solution

$$f(x) = x^3 - 8x^2 + 17x - 10$$

here three changes are observed from +ve to -ve, -ve to +ve, +ve to -ve
hence according to deccarte rule of signs $f(x)$ may have three positive roots
rewriting eq as $x(x^2 + 17) = (8x^2 + 10)$

and squaring on both sides and putting $x^2 = y$

$$y(y+17)^2 = (8y+10)^2$$

$$y^3 + 34y^2 + 289 = 64y^2 + 160y + 100$$

$$y(y^2 + 129) = 30y^2 + 100$$

putting $y^2 = z$ we get

$$z(z+129)^2 = (30z+100)^2$$

$$z^3 + 258z^2 + 16641z = 900z^2 + 6000z + 10000$$

$$z(z^2 + 16641) = (642z^2 + 10000)$$

squaring again and putting $z^2 = u$ we get

$$u(u+16641)^2 = (642u+10000)^2$$

$$u^3 + 33282u^2 + 276922881u = 412164u^2 + 12840000u + 10^8$$

$$u^3 - 378882u^2 + 264082u - 10^8 = 0$$

if the roots of the eq1 are p_1, p_2, p_3 , and those of equation 5 are q_1, q_2, q_3

$$p_1 = (q_1)^{1/8} = (-\lambda_1)^{1/8} = (378882)^{1/8} = 4.9809593 \cong 5$$

$$p_2 = (q_2)^{1/8} = (-\lambda_2 / \lambda_1)^{1/8} = (264082 / 378882)^{1/8} = 0.9558821 \cong 1$$

$$p_3 = (q_3)^{1/8} = (-\lambda_3 / \lambda_2)^{1/8} = (10^8 / 264082)^{1/8} = 2.1003064 \cong 2$$

$$\text{here } f(5) = f(2) = f(1) = 0$$

hence all are the roots of the equation

Example

Find all the root of the equation $x^4 - 3x + 1 = 0$

Solution



$$f(x) = x^4 - 3x + 1 \quad (1)$$

here there are two changes in the sign so there are two positive roots of the equation and there are no change in the roots of the equation $f(-x)$ so two positive real roots and two are complex roots

rewriting the above equation as

$$x^4 + 1 = 3x$$

squaring the equation and putting $x^2 = y$

we have

$$(y^2 + 1)^2 = 9y$$

squaring again and putting $y^2 = z$

$$(z + 1)^4 = 81z$$

$$z^4 + 4z^3 + 6z^2 - 77z + 1 = 0$$

$$z^4 + 6z^2 + 1 = -z(4z^2 - 77) \quad (2)$$

squaring once again and putting $z^2 = u$, we get

$$(u^2 + 6u + 1) = u(4u - 77)^2$$

$$u^4 - 4u^3 + 645u^2 - 5917u + 1 = 0 \quad (3)$$

if p_1, p_2, p_3, p_4 are the roots of the equation 1 and q_1, q_2, q_3, q_4 are roots of equation 3 then

$$p_1 = (q_1)^{1/8} = (-\lambda_1)^{1/8} = (4)^{1/8} = 1.189207$$

$$p_2 = (q_2)^{1/8} = \left[\frac{-\lambda_2}{\lambda_1} \right]^{1/8} = \left[\frac{654}{4} \right]^{1/8} = 1.8909921$$

$$p_3 = (q_3)^{1/8} = \left[\frac{-\lambda_3}{\lambda_2} \right]^{1/8} = \left[\frac{5917}{654} \right]^{1/8} = 1.3169384$$

$$p_4 = (q_4)^{1/8} = \left[\frac{-\lambda_4}{\lambda_3} \right]^{1/8} = \left[\frac{1}{5917} \right]^{1/8} = 0.3376659$$

from equation (2) and (3)

from equation 2 and 3, we observe the magnitude of the coefficients λ_1 and λ_2 have become constant which implies p_1 and p_4 are the real roots, then p_2 and p_3 are real roots,

let the complex roots be $\rho_2 e^{\pm i\phi_2} = \xi_2 + i\eta_2$

from equation (3) it's magnitude is given by

$$\rho_2^{2(2^3)} = \frac{\lambda_3}{\lambda_1} = \frac{5917}{4} \therefore \rho_2 = 1.5780749$$

also from equation 1 sum of roots is zero i.e. $p_1 + 2\xi_2 + p_4 = 0$

$$\xi_2 = -1/2(p_1 + p_4) = -0.7634365$$

$$\eta_2 = \sqrt{\rho_2^2 - \xi_2^2} = \sqrt{1.9074851} = 1.3811173$$

hence, the four roots are 1.1892071, 0.3376659, -0.734365 and $\pm 1.3811173i$

Revision Example

Obtain the Newton-Raphson extended formula

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2} \frac{[f(x_0)]^2}{[f'(x_0)]^2} f''(x_0)$$

For finding the root of the equation $f(x)=0$

Solution

Expanding $f(x)$ by Taylor's series and retaining up to second order term,

$$0 = f(x) = f(x_0) + (x - x_0) f'(x_0)$$

$$+ \frac{(x - x_0)^2}{2} f''(x_0)$$

Therefore,

$$f(x_1) = f(x_0) + (x_1 - x_0) f'(x_0)$$

$$+ \frac{(x_1 - x_0)^2}{2} f''(x_0) = 0$$

This can also be written as

$$f(x_0) + (x_1 - x_0) f'(x_0) + \frac{1}{2} \frac{[f'(x_0)]^2}{f''(x_0)} f''(x_0) = 0$$

Thus, the Newton-Raphson extended formula is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2} \frac{[f'(x_0)]^2}{[f''(x_0)]^3} f''(x_0)$$

This is also known as Chebyshev's formula of third order

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Solution of Linear System of Equations and Matrix Inversion

The solution of the system of equations gives n unknown values $x_1, x_2, \dots, x_n$, which satisfy the system simultaneously. If $m > n$, the system usually will have an infinite number of solutions.

If $|A| \neq 0$ then the system will have a unique solution.

If $|A| = 0$,

Then there exists no solution.

Numerical methods for finding the solution of the system of equations are classified as direct and iterative methods. In direct methods, we get the solution of the system after performing all the steps involved in the procedure. The direct methods consist of elimination methods and decomposition methods.

Under elimination methods, we consider, Gaussian elimination and Gauss-Jordan elimination methods

Crout's reduction also known as Cholesky's reduction is considered under decomposition methods.

Under iterative methods, the initial approximate solution is assumed to be known and is improved towards the exact solution in an iterative way. We consider Jacobi, Gauss-Seidel and relaxation methods under iterative methods.

Gaussian Elimination Method

In this method, the solution to the system of equations is obtained in two stages.

- i) the given system of equations is reduced to an equivalent upper triangular form using elementary transformations
- ii) the upper triangular system is solved using back substitution procedure

This method is explained by considering a system of n equations in n unknowns in the form as follows

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned} \right\}$$

Stage I: Substituting the value of x_1 from first equation into the rest

$$\left. \begin{aligned} x_1 + a'_{12}x_2 + \dots + a'_{1n}x_n &= b'_1 \\ a'_{22}x_2 + \dots + a'_{2n}x_n &= b'_2 \\ &\vdots \\ a'_{n2}x_2 + \dots + a'_{nn}x_n &= b'_n \end{aligned} \right\}$$

Now, the last $(n - 1)$ equations are independent of x_1 , that is, x_1 is eliminated from the last $(n - 1)$ equations.

This procedure is repeated and x_2 is eliminated from 3rd, 4th, ..., n -th equations

The same procedure is repeated till the given system assumes the upper triangular form:

$$\left. \begin{aligned} c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n &= d_1 \\ c_{22}x_2 + \cdots + c_{2n}x_n &= d_2 \\ &\vdots \\ c_{nn}x_n &= d_n \end{aligned} \right\}$$

Stage II: The values of the unknowns are determined by backward substitution.

First x_n is found from the last equation and then substitution this value of x_n in the preceding equation will give the value of x_{n-1} . Continuing this way, we can find the values of all other unknowns

Example

Solve the following system of equations using Gaussian elimination method

$$2x + 3y - z = 5$$

$$4x + 4y - 3z = 3$$

$$-2x + 3y - z = 1$$

Solution

Stage I (*Reduction to upper-triangular form*):

Divide first equation by 2 and then subtract the resulting equation (multiplied by 4 and -2) from the 2nd and 3rd equations respectively. Thus, we eliminate x from the 2nd and 3rd equations.

The resulting new system is given by

$$\left. \begin{aligned} x + \frac{3}{2}y - \frac{z}{2} &= \frac{5}{2} \\ 6y - 2z &= 6 \end{aligned} \right\}$$

Now, we divide the 2nd equation by -2 and eliminate y from the last equation and the modified system is given by

$$\left. \begin{array}{r} x + \frac{3}{2}y - \frac{z}{2} = \frac{5}{2} \\ y + \frac{z}{2} = 7 \\ -5z = -15 \end{array} \right\}$$

Stage II (Backward substitution):

From the last equation, we get

$$z = 3$$

Using this value of z , the second equation gives

$$y = \frac{7}{2} - \frac{3}{2} = 2$$

Putting these values of y and z in the first equation, we get

$$x = \frac{5}{2} + \frac{3}{2} - 3 = 1$$

Thus, the solution of the given system is given by

$$x = 1, y = 2, z = 3$$

Partial and Full Pivoting

The Gaussian elimination method fails if any one of the pivot elements becomes zero. In such a situation, we rewrite the equations in a different order to avoid zero pivots. Changing the order of equations is called pivoting.

Partial pivoting

If the pivot happens to be zero, then the i -th column elements are searched for the numerically largest element. Let the j -th row ($j > i$) contains this element, then we interchange the i -th equation with the j -th equation and proceed for elimination. This process is continued whenever pivots become zero during elimination.

For example, let us examine the solution of the following simple system

$$10^{-5}x_1 + x_2 = 1$$

$$x_1 + x_2 = 2$$

Using Gaussian elimination method with and without partial pivoting, assuming that we require the solution accurate to only four decimal places. The solution by Gaussian elimination gives $x_1 = 0, x_2 = 1$.

If we use partial pivoting, the system takes the form

$$\begin{aligned}x_1 + x_2 &= 2 \\ 10^{-5}x_1 + x_2 &= 1\end{aligned}$$

Using Gaussian elimination method, the solution is found to be $x_1 = 1$, $x_2 = 1$, which is a meaningful and perfect result.

In full pivoting (or complete pivoting), we interchange rows and columns, such that the largest element in the matrix of the variables also get changed. Full pivoting, in fact, is more complicated than the partial pivoting. Partial pivoting is preferred for hand computation.

The general form of a system of m linear equations in n unknowns $x_1, x_2, x_3, \dots, x_n$ can be represented in matrix form as under:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Using matrix notation, the above system can be written in compact form as $[A](X) = (B)$

Note:

1. This method fails if any of the pivots become zero in such cases, by interchanging the rows we can get the non-zero pivots.

Example

Solve the system of equations by Gaussian elimination method with partial pivoting.

$$x + y + z = 7$$

$$3x + 3y + 4z = 24$$

$$2x + y + 3z = 16$$

Solution

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 24 \\ 16 \end{bmatrix}$$

To start with, we observe that the pivot element $a_{11} = 1 (\neq 0)$.

However, a glance at the first column reveals that the numerically largest element is 3 which is in second row. Hence $R1 \leftrightarrow R2$

Thus the given equation takes the form after partial pivoting

$$\begin{bmatrix} 3 & 3 & 4 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 24 \\ 7 \\ 16 \end{bmatrix}$$

Stage I (Reduction to upper triangular form):

$$\begin{bmatrix} 1 & 1 & \frac{4}{3} \\ 0 & -1 & \frac{1}{3} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix}$$

Stage II (Back substitution):

$$z = 3$$

$$-y + 1 = 0 \text{ or } y = 1$$

$$x + 1 + 4 = 8 \text{ or } x = 3$$

Example

Solve the following system of equations by Gaussian elimination method with partial pivoting

$$0x_1 + 4x_2 + 2x_3 + 8x_4 = 24$$

$$4x_1 + 10x_2 + 5x_3 + 4x_4 = 32$$

$$4x_1 + 5x_2 + 6.5x_3 + 2x_4 = 26$$

$$9x_1 + 4x_2 + 4x_3 + 0x_4 = 21$$

Solution

In matrix notation, the given system can be written as

$$\begin{bmatrix} 0 & 4 & 2 & 8 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 6.5 & 2 \\ 9 & 4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 24 \\ 32 \\ 26 \\ 21 \end{bmatrix}$$

To start with, we observe that the pivot row, that is, the first row has a zero pivot element ($a_{11} = 0$). This row should be interchanged with any row following it, which on becoming a pivot row should not have a zero pivot element.

While interchanging rows it is better to interchange the first and fourth rows, which is called partial pivoting and get,

$$\begin{bmatrix} 9 & 4 & 4 & 0 \\ 4 & 10 & 5 & 4 \\ 4 & 5 & 6.5 & 2 \\ 0 & 4 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 21 \\ 32 \\ 26 \\ 24 \end{bmatrix}$$

Stage I (Reduction to upper-triangular form):

$$\begin{bmatrix} 1 & 4 & 4 & 0 \\ 0 & 8.2222 & 3.2222 & 4 \\ 0 & 3.2222 & 4.7222 & 2 \\ 0 & 4 & 2 & 8 \end{bmatrix} \begin{bmatrix} x \\ x_2^1 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.333 \\ 22.6666 \\ 16.6666 \\ 24 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 4 & 0 \\ 0 & 1 & 0.3919 & 0.4865 \\ 0 & 0 & 3.4594 & 0.4324 \\ 0 & 0 & 0.4324 & 6.0540 \end{bmatrix} \begin{bmatrix} x \\ x_2^1 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.333 \\ 2.7568 \\ 7.78836 \\ 12.9728 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 4 & 0 \\ 0 & 1 & 0.3919 & 0.4865 \\ 0 & 0 & 1 & 0.1250 \\ 0 & 0 & 0 & 5.999 \end{bmatrix} \begin{bmatrix} x \\ x_2^1 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.3333 \\ 2.7568 \\ 2.2500 \\ 11.9999 \end{bmatrix}$$

Stage II

Back substitution

$$x_1 = 1.0, \quad x_2 = 1.0, \quad x_3 = 2.0, \quad x_4 = 2.0$$

Example

$$3x + y - z = 3$$

Solve the system of equations $2x - 8y + z = -5$ using Gauss elimination method.

$$x - 2y + 9z = 8$$

Solution

The given system is equivalent to

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & -8 & 1 \\ 1 & -2 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 8 \end{bmatrix}$$

$$A \quad X = B$$

the augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 1 & -1 & 3 \\ 2 & -8 & 1 & -5 \\ 1 & -2 & 9 & 8 \end{array} \right]$$

now making A as an upper triangular matrix

$$\sim \left[\begin{array}{ccc|c} 3 & 1 & -1 & 3 \\ 0 & -\frac{26}{3} & 5 & -7 \\ 0 & -\frac{7}{3} & \frac{28}{3} & 7 \end{array} \right] \quad R_2 - \frac{2}{3}R_1, R_3 - \frac{1}{3}R_1$$

now choosing $-\frac{26}{3}$ as the pivot from the second column,

$$\sim \left[\begin{array}{ccc|c} 3 & 1 & -1 & 3 \\ 0 & -\frac{26}{3} & 5 & -7 \\ 0 & 0 & \frac{693}{78} & \frac{231}{26} \end{array} \right] \quad R_2 - \frac{2}{3}R_1, R_3 - \frac{1}{3}R_1$$

from this we get

$$3x + y - z = 3$$

$$-\frac{26}{3}y + \frac{5}{3}z = -7$$

$$\frac{693}{78}z = \frac{231}{26}$$

now by back substitution $z = 1$

$$-\frac{26}{3}y = -7 - \frac{5}{3}(z) = -7 - \frac{5}{3}(1) = \frac{-26}{3} \Rightarrow y = 1$$

$$\text{now } x = \frac{1}{3}[3 - y + z] = \frac{1}{3}[3 - 1 + 1] = 1$$

so the solution is $x = y = z = 1$

Example

Using Gauss elimination method, solve the system.

$$3.15x - 1.96y + 3.85z = 12.95$$

$$2.13x + 5.12y - 2.89z = -8.61$$

$$5.92x + 3.05y + 2.15z = 6.88$$

Solution

The given system is equivalent to

$$\begin{bmatrix} 3.15 & -1.96 & 3.85 \\ 2.13 & 5.12 & -2.89 \\ 5.92 & 3.05 & 2.15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12.95 \\ -8.61 \\ 6.88 \end{bmatrix}$$

$$A \quad X = B$$

$$[A|B] = \left[\begin{array}{ccc|c} 3.15 & -1.96 & 3.85 & 12.95 \\ 2.13 & 5.12 & -2.89 & -8.61 \\ 5.92 & 3.05 & 2.15 & 6.88 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 3.15 & -1.96 & 3.85 & 12.95 \\ 0 & 6.4453 & -5.4933 & -17.3667 \\ 0 & 6.7335 & -5.0855 & -17.4578 \end{array} \right] \quad \begin{array}{l} R_2 - \frac{2.13}{3.15} R_1, R_3 - \frac{5.92}{3.15} R_1 \\ \\ R_3 - \frac{6.7335}{6.4453} R_2 \end{array}$$

choosing 6.4453 as pivot

$$\sim \left[\begin{array}{ccc|c} 3.15 & -1.96 & 3.85 & 12.95 \\ 0 & 6.4453 & -5.4933 & -17.3667 \\ 0 & 0 & 0.6534 & 0.6853 \end{array} \right]$$

from this, we get

$$3.15x - 1.96y + 3.85z = 12.95$$

$$6.4453y - 5.4933z = -17.3667$$

$$0.6534z = 0.6853$$

by backward substitution

$$z = \frac{0.6853}{0.6534} = 1.0488215$$

$$y = \frac{5.4933 - 17.3667}{6.4453} = -1.8005692$$

$$x = \frac{1.96y - 3.85z + 12.95}{3.15} = 1.708864$$

Example

Solve the system of equation

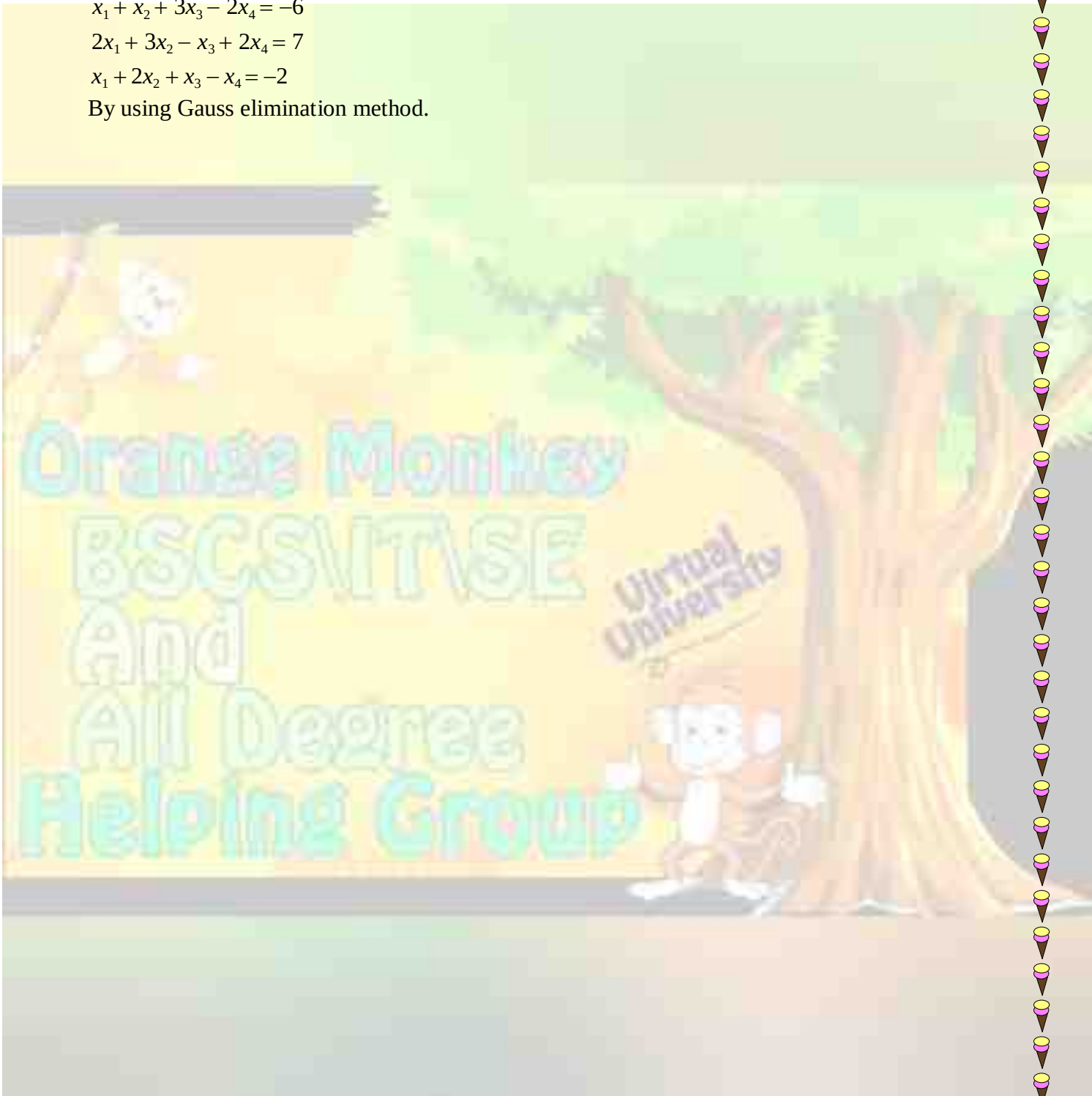
$$x_1 + x_2 + x_3 + x_4 = 2$$

$$x_1 + x_2 + 3x_3 - 2x_4 = -6$$

$$2x_1 + 3x_2 - x_3 + 2x_4 = 7$$

$$x_1 + 2x_2 + x_3 - x_4 = -2$$

By using Gauss elimination method.



$$x_1 + x_2 + x_3 + x_4 = 2$$

$$x_1 + x_2 + 3x_3 - 2x_4 = -6$$

$$2x_1 + 3x_2 - x_3 + 2x_4 = 7$$

$$x_1 + 2x_2 + x_3 - x_4 = -2$$

the given system in matrix form is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & -2 \\ 2 & 3 & -1 & 2 \\ 1 & 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 7 \\ -2 \end{bmatrix}$$

$$A \quad X = B$$

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 3 & -2 & -6 \\ 2 & 3 & -1 & 2 & 7 \\ 1 & 2 & 1 & -1 & -2 \end{array} \right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & -3 & -8 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 1 & 0 & -2 & -4 \end{array} \right] \begin{matrix} \\ R_2 - R_1, R_3 - 2R_1, R_4 - R_1 \end{matrix}$$

since the element in the second column is zero so interchanging the rows

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 0 & 2 & -3 & -8 \\ 0 & 1 & 0 & -2 & -4 \end{array} \right] \begin{matrix} \\ R_{23} \end{matrix}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 0 & 2 & -3 & -8 \\ 0 & 1 & 3 & -2 & -4 \end{array} \right] \begin{matrix} \\ R_4 - R_2 \end{matrix}$$

now the pivot is 2, therefore

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 0 & 3 \\ 0 & 0 & 2 & -3 & -8 \\ 0 & 0 & 0 & 5 & 5 \end{array} \right] \begin{matrix} \\ R - \frac{3}{2}R \end{matrix}$$

from this we get

$$x_1 + x_2 + x_3 + x_4 = 2$$

$$x_2 - 3x_3 = 3$$

$$2x_3 - 3x_4 = -8$$

$$\left(\frac{5}{2}\right)x_4 = 5$$

$$\text{now } x_4 = 2$$

$$x_3 = \frac{1}{2}(-8 + 3x_4) = \frac{1}{2}(-8 + 6) = -1$$

now

$$x_2 = 3 + 3x_3 = 3 - 3 = 0$$

now from equation 1

$$x_1 = 2 - x_2 - x_3 - x_4 = 2 - 0 + 1 - 2 = 1$$

$$\text{so } x_1 = 1, x_2 = 0, x_3 = -1, x_4 = 2$$

Solution of Linear System of Equations and Matrix Inversion

Gauss–Jordan Elimination Method

This method is a variation of Gaussian elimination method. In this method, the elements above and below the diagonal are simultaneously made zero. That is a given system is reduced to an equivalent diagonal form using elementary transformations. Then the solution of the resulting diagonal system is obtained. Sometimes, we normalize the pivot row with respect to the pivot element, before elimination. Partial pivoting is also used whenever the pivot element becomes zero.

Example

Solve the system of equations using Gauss-Jordan elimination method:

$$\begin{cases} x + 2y + z = 8 \\ 2x + 3y + 4z = 20 \\ 4x + 3y + 2z = 16 \end{cases}$$

Solution

In matrix notation, the given system can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 20 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & -5 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ -16 \end{bmatrix} \quad (-2) R1 + R2 \text{ and } (-4) R1 + R3$$

Now, we eliminate y from the first and third rows using the second row. Thus, we get

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \\ -36 \end{bmatrix}$$

Before, eliminating z from the first and second row, normalize the third row with respect to the pivot element, we get

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \\ 3 \end{bmatrix}$$

Using the third row of above Equation, and eliminating z from the first and second rows, we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

The solution is

$$x = 1, y = 2, z = 3.$$

Example

Solve the system of equation by using the Gauss Jordan elimination method

$$10x + y + z = 12$$

$$2x + 10y + z = 13$$

$$x + y + 5z = 7$$

Solution

$$10x + y + z = 12$$

$$2x + 10y + z = 13$$

$$x + y + 5z = 7$$

the given system

$$\begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 7 \end{bmatrix}$$

the augmented matrix may be written as

$$[A/B] = \left[\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -8 & -44 & -51 \\ 2 & 10 & 1 & 13 \\ & & & \end{array} \right] \begin{matrix} \\ R - 9R \\ 1 \quad 3 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 1 & -8 & -44 & -51 \\ 0 & 26 & 89 & 115 \end{array} \right] \begin{matrix} \\ R - 2R, R - R \\ 2 \quad 1 \quad 3 \quad 1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 0 & 9 & 49 & 58 \\ 1 & -8 & -44 & -51 \\ 0 & 1 & 89 & 59 \end{array} \right] \begin{matrix} \\ R - 3R \\ 2 \quad 3 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 0 & 9 & 49 & 58 \\ 1 & 0 & 420 & 421 \\ 0 & 1 & 58 & 59 \end{array} \right] \begin{matrix} \\ R + 8R, R - 9R \\ 1 \quad 2 \quad 3 \quad 2 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 0 & 0 & -473 & -473 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{array} \right] \begin{matrix} \\ \frac{1}{473}R, R - 420R, R - 58R \\ 3 \quad 1 \quad 3 \quad 2 \quad 2 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{array} \right]$$

thus the system reduces to reduced echlon form

$$\text{so } x = y = z = 1$$

Example

Solve the system of equations by Gauss Jordan method

$$10x_1 + x_2 + x_3 = 12$$

$$x_1 + 10x_2 - x_3 = 10$$

$$x_1 - 2x_2 + 10x_3 = 9$$

Solution

$$10x_1 + x_2 + x_3 = 12$$

$$x_1 + 10x_2 - x_3 = 10$$

$$x_1 - 2x_2 + 10x_3 = 9$$

the matrix form of the given system may be written as

$$\begin{bmatrix} 10 & 1 & 1 \\ 1 & 10 & -1 \\ 1 & -2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \\ 9 \end{bmatrix}$$

the augmented matrix may be written as

$$[A/B] = \left[\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 1 & 10 & -1 & 10 \\ 1 & -2 & 10 & 9 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -89 & 10 & -78 \\ 1 & 10 & -1 & 88 \\ 1 & -2 & 10 & 87 \end{array} \right] \begin{matrix} \\ R - 9R \\ \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -89 & 10 & -78 \\ 0 & 9 & -1 & 8 \\ 0 & 1 & 0 & 1 \end{array} \right] \begin{matrix} \\ R - R, R - R \\ \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -89 & 10 & -78 \\ 0 & 9 & -1 & 8 \\ 0 & 1 & 0 & 1 \end{array} \right] \begin{matrix} \\ R_2 \cdot R_3 \\ \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -89 & 10 & -78 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{array} \right] \begin{matrix} \\ R - 8R \\ \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -79 & -78 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right] \begin{matrix} \\ R - 89R, R - R \\ \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right] \begin{matrix} \\ R + 79R, R + R \\ \end{matrix}$$

so the system gives the values of all the three variables

$$x_1 = x_2 = x_3 = 1$$

Example

Solve the system of equations by using Gauss Jordan method.

$$x + 2y + z - w = -2$$

$$2x + 3y - z + 2w = 7$$

$$x + y + 3z - 2w = -6$$

$$x + y + z + w = 2$$

Solution

$$x + 2y + z - w = -2$$

$$2x + 3y - z + 2w = 7$$

$$x + y + 3z - 2w = -6$$

$$x + y + z + w = 2$$

the system may be written as

$$\begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 3 & -1 & 2 \\ 1 & 1 & 3 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ -6 \\ 2 \end{bmatrix}$$

the augmented matrix may be written as

$$[A/B] = \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & -2 \\ 2 & 3 & -1 & 2 & 7 \\ 1 & 1 & 3 & -2 & -6 \\ 1 & 1 & 1 & 1 & 2 \end{array} \right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & -2 \\ 0 & 1 & 3 & -4 & 11 \\ 0 & -1 & 2 & -1 & -4 \\ 0 & -1 & 0 & 2 & 4 \end{array} \right] \quad R_2 - 2R_1, R_3 - R_1, R_4 - R_1$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -5 & 7 & 20 \\ 0 & 1 & 3 & -4 & -11 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 3 & -2 & -7 \end{array} \right] \quad R_1 - 2R_2, \frac{1}{5}(R_3 + R_2), R_4 + R_2$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right] \quad R_1 + 5R_3, R_2 - 3R_3, R_4 - 3R_3$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right] \quad R_1 - 2R_4, R_2 + R_4, R_3 + R_4$$

the system may be written as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

the values of all the variables are

$$x = 1, y = 0, z = -1, w = 2$$

Crout's Reduction Method

Here the coefficient matrix $[A]$ of the system of equations is decomposed into the product of two matrices $[L]$ and $[U]$, where $[L]$ is a lower-triangular matrix and $[U]$ is an upper-triangular matrix with 1's on its main diagonal.

For the purpose of illustration, consider a general matrix in the form

$$[L][U] = [A]$$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The sequence of steps for getting [L] and [U] are given below:

Step I: Multiplying all the rows of [L] by the first column of [U], we get

$$l_{11} = a_{11}, \quad l_{21} = a_{21}, \quad l_{31} = a_{31}$$

Thus, we observe that the first column of [L] is same as the first column of [A].

Step II: Multiplying the first row of [L] by the second and third columns of [U], we obtain

$$l_{11}u_{12} = a_{12}, \quad l_{11}u_{13} = a_{13}$$

Or

$$u_{12} = \frac{a_{12}}{l_{11}}, \quad u_{13} = \frac{a_{13}}{l_{11}}$$

Thus, the first row of [U] is obtained. We continue this process, getting alternately the column of [L] and a row of [U].

Step III: Multiply the second and third rows of [L] by the second column of [U] to get

$$l_{21}u_{12} + l_{22} = a_{22}, \quad l_{31}u_{12} + l_{32} = a_{32}$$

This gives

$$l_{22} = a_{22} - l_{21}u_{12}, \quad l_{32} = a_{32} - l_{31}u_{12}$$

Step IV: Now, multiply the second row of [L] by the third column of [U] which yields

$$l_{21}u_{13} + l_{22}u_{23} = a_{23}$$

$$u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}}$$

Step V: Lastly, we multiply the third row of [L] by the third column of [U] and get

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = a_{33}$$

This gives

$$l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$$

Thus, the above five steps determine [L] and [U].

This algorithm can be generalized to any linear system of order n.

Consider a system of equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

In matrix notation as $[A](X) = (B)$. Let $[A] = [L][U]$, then we get,

$$[L][U](X) = (B)$$

Substituting $[U](X) = (Z)$ in Eq. we obtain $[L](Z) = (B)$

$$\begin{cases} l_{11}z_1 = b_1 \\ l_{21}z_1 + l_{22}z_2 = b_2 \\ l_{31}z_1 + l_{32}z_2 + l_{33}z_3 = b_3 \end{cases}$$

Having computed z_1, z_2 and z_3 , we can compute x_1, x_2 , and x_3 from equation $[U](X) = (Z)$ or from

$$\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

This method is also known as Cholesky reduction method. This technique is widely used in the numerical solutions of partial differential equations.

This method is very popular from computer programming point of view, since the storage space reserved for matrix [A] can be used to store the elements of [L] and [U] at the end of computation.

This method fails if any

$$a_{ii} = 0$$

Example

Solve the following system of equations by Crout's reduction method

$$5x_1 - 2x_2 + x_3 = 4$$

$$7x_1 + x_2 - 5x_3 = 8$$

$$3x_1 + 7x_2 + 4x_3 = 10$$

Solution

Let the coefficient matrix [A] be written as [L] [U]. Thus,

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

Step I: Multiply all the rows of [L] by the first column of [U], we get

$$l_{11} = 5, \quad l_{21} = 7, \quad l_{31} = 3$$

Step II: Multiply the first row of [L] by the second and third columns of [U], we have

$$l_{11}u_{12} = -2, \quad l_{11}u_{13} = 1$$

$$u_{12} = -\frac{2}{5}, \quad u_{13} = \frac{1}{5}$$

STEP III: Multiply the 2nd and 3rd rows of [L] by the 2nd column of [U], we get

$$\begin{aligned} l_{21}u_{12} + l_{22} &= 1 \quad \text{or} \quad l_{22} = 1 + \frac{14}{5} = \frac{19}{5} \\ l_{31}u_{12} + l_{32} &= 7 \quad \text{or} \quad l_{32} = 7 + \frac{19}{5} = \frac{41}{5} \end{aligned}$$

STEP IV: Multiply the 2nd row of [L] by the 3rd column of [U]

$$l_{21}u_{13} + l_{22}u_{23} = -5$$

$$\frac{19}{5}u_{23} = -5 - \frac{7}{5}$$

$$u_{23} = -\frac{32}{19}$$

STEP V: Finally, multiply the 3rd row of [L] with the 3rd column of [U], we obtain

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 4$$

$$l_{33} = \frac{327}{19}$$

Thus, the given system of equations takes the form [L][U][X] = (B).

$$\begin{bmatrix} 5 & 0 & 0 \\ 7 & \frac{19}{5} & 0 \\ 3 & \frac{41}{5} & \frac{327}{19} \end{bmatrix} \begin{bmatrix} 1 & -\frac{2}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{32}{19} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$

That is,

$$\begin{bmatrix} 5 & 0 & 0 \\ 7 & \frac{5}{41} & 0 \\ 3 & \frac{5}{19} & \frac{1}{327} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$

Let $[U](X) = (Z)$, then

$$[L](Z) = (4 \ 8 \ 10)^T$$

Or

$$\begin{bmatrix} 5 & 0 & 0 \\ 7 & \frac{5}{41} & 0 \\ 3 & \frac{5}{19} & \frac{1}{327} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$

Which gives utilizing these values of z , the Eq becomes

By back substitution method, we obtain

$$x_3 = \frac{46}{327}, \quad x_2 = \frac{284}{327}, \quad x_1 = \frac{366}{327}$$

This is the required solution.

Example

Solve the following system

$$\begin{aligned} 2x - 3y + 10z &= 3 \\ -x + 4y + 2z &= 20 \\ 5x + 2y + z &= -12 \end{aligned}$$

Solution

$$2x - 3y + 10z = 3$$

$$-x + 4y + 2z = 20$$

$$5x + 2y + z = -12$$

the given system is $AX = B$

$$A = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

let $LU = A$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}$$

here

$$u_{11} = 2, u_{12} = -3, u_{13} = 10$$

$$l_{21} u_{11} = -1 \Rightarrow l_{21} = \frac{-1}{2}$$

$$l_{31} u_{11} = 5 \Rightarrow l_{31} = \frac{5}{2}$$

$$l_{21} u_{12} + u_{22} = 2 \Rightarrow u_{22} = 2 - l_{21} u_{12} = 2 - \left(\frac{-1}{2}\right)(-3) = \frac{5}{2}$$

$$l_{31} u_{12} + l_{32} u_{22} + u_{33} = 5 \Rightarrow u_{33} = 5 - l_{31} u_{12} - l_{32} u_{22} = 5 - \left(\frac{5}{2}\right)\left(\frac{5}{2}\right) = \frac{-25}{4}$$

$$l_{21} u_{13} + l_{22} u_{23} + u_{24} = 2 \Rightarrow u_{24} = 2 - l_{21} u_{13} - l_{22} u_{23} = 2 - \left(\frac{-1}{2}\right)(10) - \left(\frac{5}{2}\right)\left(\frac{5}{2}\right) = \frac{-25}{4}$$

$$l_{31} u_{13} + l_{32} u_{23} + l_{33} u_{33} + u_{34} = 5 \Rightarrow u_{34} = 5 - l_{31} u_{13} - l_{32} u_{23} - l_{33} u_{33} = \frac{253}{5}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{5}{2} & \frac{19}{5} & 1 \end{bmatrix}, U = \begin{bmatrix} 2 & -3 & 10 \\ 0 & \frac{5}{2} & 7 \\ 0 & 0 & \frac{-253}{5} \end{bmatrix}$$

let $UX = Y$ where $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, then $LY = B$,

$$\text{i.e.} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{5}{2} & \frac{19}{5} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -20 \\ -12 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{5}{2} & \frac{19}{5} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

now eqn (1) implies

$$y_1 = 3$$

$$-\frac{1}{2}y_1 + y_2 = -20 \Rightarrow y_2 = \frac{43}{2}$$

$$\frac{5}{2}y_1 + \frac{19}{5}y_2 + y_3 = -12 \Rightarrow y_3 = \frac{-506}{5}$$

$$2x - 3y + 10z = 3, \frac{5}{2}y + 7z = \frac{43}{2}, -\frac{253}{5}z = \frac{-506}{3}$$

by back substitution $x = -4, y = 3$ and $z = 2$

Example

Solve the following system

$$x + 3y + 4z = 4$$

$$x + 4y + 3z = -2$$

$$x + 3y + 4z = 1$$

Solution

$$x + 3y + 4z = 4$$

$$x + 4y + 3z = -2$$

$$x + 3y + 4z = 1$$

the given system is $AX = B$

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

let $LU = A$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

here

$$u_{11} = 1, u_{12} = 3, u_{13} = 8$$

$$l_{21}u_{11} = -1 \Rightarrow l_{21} = -1$$

$$l_{31}u_{11} = 1 \Rightarrow l_{31} = 1$$

$$l_{21}u_{13} + u_{23} = 2 \Rightarrow u_{23} = 4 - l_{21}u_{13} = 4 - (1)3 = 1$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} \Rightarrow u_{33} = 4 - l_{31}u_{13} - l_{32}u_{23} = -4$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & -4 \end{bmatrix}$$

let $UX = Y$ where $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, then $LY = B$,

$$\text{i.e. } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 1 & 0 \\ 5 & 19 & 1 \\ 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

now eqn (1) implies

$$y_1 = 4$$

$$y_2 = -2$$

$$y_1 + y_3 = 1 \Rightarrow y_3 = 1 - y_1 = -3$$

we also have

$$x + 3y + 8z = 4$$

$$y - 5z = -2$$

$$-4z = -3$$

by back substituton

$$z = \frac{3}{4}$$

$$y = -2 + 5z = -2 + 5\left(\frac{3}{4}\right) = \frac{7}{4}$$

$$x = 4 - 3y - 8z = 4 - 3\left(\frac{7}{4}\right) - 8\left(\frac{3}{4}\right) = \frac{29}{4}$$

Solution of Linear System of Equations and Matrix Inversion

Jacobi's Method

This is an iterative method, where initial approximate solution to a given system of equations is assumed and is improved towards the exact solution in an iterative way.

In general, when the coefficient matrix of the system of equations is a sparse matrix (many elements are zero), iterative methods have definite advantage over direct methods in respect of economy of computer memory.

Such sparse matrices arise in computing the numerical solution of partial differential equations.

Let us consider

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \right\}$$

In this method, we assume that the coefficient matrix $[A]$ is strictly diagonally dominant, that is, in each row of $[A]$ the modulus of the diagonal element exceeds the sum of the off-diagonal elements.

We also assume that the diagonal element do not vanish. If any diagonal element vanishes, the equations can always be rearranged to satisfy this condition.

Now the above system of equations can be written as

$$\left. \begin{aligned} x_1 &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2 - \cdots - \frac{a_{1n}}{a_{11}}x_n \\ x_2 &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1 - \cdots - \frac{a_{2n}}{a_{22}}x_n \\ \vdots &\vdots \\ x_n &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1 - \cdots - \frac{a_{nn-1}}{a_{nn}}x_{n-1} \end{aligned} \right\}$$

We shall take this solution vector $(x_1, x_2, \dots, x_n)^T$ as a first approximation to the exact solution of system. For convenience, let us denote the first approximation vector by $(x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)})$ got after taking as an initial starting vector.

Substituting this first approximation in the right-hand side of system, we obtain the second approximation to the given system in the form

$$\left. \begin{aligned} x_1^{(2)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(1)} - \dots - \frac{a_{1n}}{a_{11}} x_n^{(1)} \\ x_2^{(2)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(1)} - \dots - \frac{a_{2n}}{a_{22}} x_n^{(1)} \\ &\vdots \\ x_n^{(2)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(1)} - \dots - \frac{a_{nn-1}}{a_{nn}} x_{n-1}^{(1)} \end{aligned} \right\}$$

This second approximation is substituted into the right-hand side of Equations and obtain the third approximation and so on.

This process is repeated and $(r+1)th$ approximation is calculated

$$\left. \begin{aligned} x_1^{(r+1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(r)} - \dots - \frac{a_{1n}}{a_{11}} x_n^{(r)} \\ x_2^{(r+1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(r)} - \dots - \frac{a_{2n}}{a_{22}} x_n^{(r)} \\ &\vdots \\ x_n^{(r+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(r)} - \dots - \frac{a_{nn-1}}{a_{nn}} x_{n-1}^{(r)} \end{aligned} \right\}$$

Briefly, we can rewrite these Equations as

$$x_i^{(r+1)} = \frac{b_i}{a_{ii}} - \sum_{\substack{j=1 \\ j \neq i}}^n \frac{a_{ij}}{a_{ii}} x_j^{(r)},$$

$$r = 1, 2, \dots, \quad i = 1, 2, \dots, n$$

It is also known as method of simultaneous displacements, since no element of $x_i^{(r+1)}$ is used in this iteration until every element is computed.

A sufficient condition for convergence of the iterative solution to the exact solution is

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}|, \quad i = 1, 2, \dots, n$$

When this condition (diagonal dominance) is true, Jacobi's method converges

Example

Find the solution to the following system of equations using Jacobi's iterative method for the first five iterations:

$$83x + 11y - 4z = 95$$

$$7x + 52y + 13z = 104$$

$$3x + 8y + 29z = 71$$

Solution

$$\left. \begin{aligned} x &= \frac{95}{104} - \frac{11}{83}y + \frac{4}{83}z \\ y &= \frac{83}{29} - \frac{x}{29} - \frac{z}{29} \\ z &= \frac{52}{29} - \frac{32}{29}x - \frac{82}{29}y \end{aligned} \right\}$$

Taking the initial starting of solution vector as $(0,0,0)^T$, from Eq. ,we have the first approximation as

$$\begin{pmatrix} x^{(1)} \\ y^{(1)} \\ z^{(1)} \end{pmatrix} = \begin{pmatrix} 1.1446 \\ 2.0000 \\ 2.4483 \end{pmatrix}$$

Now, using Eq. ,the second approximation is computed from the equations

$$\left. \begin{aligned} x^{(2)} &= 1.1446 - 0.1325y^{(1)} + 0.0482z^{(1)} \\ y^{(2)} &= 2.0 - 0.1346x^{(1)} - 0.25z^{(1)} \\ z^{(2)} &= 2.4483 - 0.1035x^{(1)} - 0.2759y^{(1)} \end{aligned} \right\}$$

Making use of the last two equations we get the second approximation as

$$\begin{pmatrix} x^{(2)} \\ y^{(2)} \\ z^{(2)} \end{pmatrix} = \begin{pmatrix} 0.9976 \\ 1.2339 \\ 1.7424 \end{pmatrix}$$

Similar procedure yields the third, fourth and fifth approximations to the required solution and they are tabulated as below;

Variables			
Iteration number r	x	y	z
1	1.1446	2.0000	2.4483
2	0.9976	1.2339	1.7424
3	1.0651	1.4301	2.0046

4	1.0517	1.3555	1.9435
5	1.0587	1.3726	1.9655

Example

Solve the system by jacobi's iterative method

$$8x - 3y + 2z = 20$$

$$4x + 11y - z = 33$$

$$6x + 3y + 12z = 35$$

(Perform only four iterations)

Solution

Consider the given system as

$$8x - 3y + 2z = 20$$

$$4x + 11y - z = 33$$

$$6x + 3y + 12z = 35$$

the system is diagonally dominant

$$x = \frac{1}{8}[20 + 3y - 2z]$$

$$y = \frac{1}{11}[33 - 4x + z]$$

$$z = \frac{1}{12}[35 - 6x - 3y]$$

we start with an initial approximation $x_0 = y_0 = z_0 = 0$

substituting these

first iteration

$$x_1 = \frac{1}{8}[20 + 3(0) - 2(0)] = 2.5$$

$$y_1 = \frac{1}{11}[33 - 4(0) + 0] = 3$$

$$z_1 = \frac{1}{12}[35 - 6(0) - 3(0)] = 2.916667$$

Second iteration

$$x_2 = \frac{1}{8}[20 + 3(3) - 2(2.916667)] = 2.895833$$

$$y_2 = \frac{1}{11}[33 - 4(2.5) + 2.916667] = 2.3560606$$

$$z_2 = \frac{1}{12}[35 - 6(2.5) - 3(3)] = 0.9166666$$

third iteration

$$x_3 = \frac{1}{8} [20 + 3(2.3560606) - 2(0.9166666)] = 3.1543561$$

$$y_3 = \frac{1}{11} [33 - 4(2.8958333) + 0.9166666] = 2.030303$$

$$z_3 = \frac{1}{12} [35 - 6(2.8958333) - 3(2.3560606)] = 0.8797348$$

fourth iteration

$$x_4 = \frac{1}{8} [20 + 3(2.030303) - 2(0.8797348)] = 3.0419299$$

$$y_4 = \frac{1}{11} [33 - 4(3.1543561) + 0.8797348] = 1.9329373$$

$$z_4 = \frac{1}{12} [35 - 6(3.1543561) - 3(2.030303)] = 0.8319128$$

Example

Solve the system by jacobi's iterative method

$$3x + 4y + 15z = 54.8$$

$$x + 12y + 3z = 39.66$$

$$10x + y - 2z = 7.74$$

(Perform only four iterations)

Solution

Consider the given system as

$$3x + 4y + 15z = 54.8$$

$$x + 12y + 3z = 39.66$$

$$10x + y - 2z = 7.74$$

the system is not diagonally dominant we rearrange the system

$$10x + y - 2z = 7.74$$

$$x + 12y + 3z = 39.66$$

$$3x + 4y + 15z = 54.8$$

$$x = \frac{1}{10} [7.74 - y + 2z]$$

$$y = \frac{1}{12} [39.66 - x - 3z]$$

$$z = \frac{1}{15} [54.8 - 3x - 4y]$$

we start with an initial approximation $x_0 = y_0 = z_0 = 0$

substituting these

first iteration

$$x_1 = \frac{1}{10} [7.74 - (0) + 2(0)] = 0.774$$

$$y_1 = \frac{1}{12} [39.66 - (0) - 3(0)] = 1.1383333$$

$$z_1 = \frac{1}{15} [54.8 - 3(0) - 4(0)] = 3.6533333$$

Second iteration

$$x_2 = \frac{1}{10} [7.74 - 1.1383333 + 2(3.6533333)] = 1.3908333$$

$$y_2 = \frac{1}{12} [39.66 - 0.774 - 3(3.6533333)] = 2.3271667$$

$$z_2 = \frac{1}{15} [54.8 - 3(0.774) - 4(1.1383333)] = 3.1949778$$

third iteration

$$x_3 = \frac{1}{10} [7.74 - 2.3271667 + 2(3.1949778)] = 1.1802789$$

$$y_3 = \frac{1}{12} [39.66 - 1.3908333 - 3(3.1949778)] = 2.3903528$$

$$z_3 = \frac{1}{15} [54.8 - 3(1.3908333) - 4(2.3271667)] = 2.7545889$$

fourth iteration

$$x_4 = \frac{1}{10} [7.74 - 2.5179962 + 2(2.7798501)] = 1.0781704$$

$$y_4 = \frac{1}{12} [39.66 - 1.1802789 - 3(2.7545889)] = 2.51779962$$

$$z_4 = \frac{1}{15} [54.8 - 3(1.1802789) - 4(2.3903528)] = 2.7798501$$

Solution of Linear System of Equations and Matrix Inversion

Gauss–Seidel Iteration Method

It is another well-known iterative method for solving a system of linear equations of the form

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \right\}$$

In Jacobi's method, the $(r + 1)$ th approximation to the above system is given by Equations

$$\left. \begin{aligned} x_1^{(r+1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2^{(r)} - \cdots - \frac{a_{1n}}{a_{11}}x_n^{(r)} \\ x_2^{(r+1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1^{(r)} - \cdots - \frac{a_{2n}}{a_{22}}x_n^{(r)} \\ \vdots &\vdots \\ x_n^{(r+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1^{(r)} - \cdots - \frac{a_{nn-1}}{a_{nn}}x_{n-1}^{(r)} \end{aligned} \right\}$$

Here we can observe that no element of $x_i^{(r+1)}$ replaces $x_i^{(r)}$ entirely for the next cycle of computation.

In Gauss-Seidel method, the corresponding elements of $x_i^{(r+1)}$ replaces those of $x_i^{(r)}$ as soon as they become available.

Hence, it is called the method of successive displacements. For illustration consider

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \right\}$$

In Gauss-Seidel iteration, the $(r + 1)$ th approximation or iteration is computed from:

$$\left. \begin{aligned} x_1^{(r+1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2^{(r)} - \cdots - \frac{a_{1n}}{a_{11}}x_n^{(r)} \\ x_2^{(r+1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1^{(r+1)} - \cdots - \frac{a_{2n}}{a_{22}}x_n^{(r)} \\ \vdots &\vdots \\ x_n^{(r+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1^{(r+1)} - \cdots - \frac{a_{nn-1}}{a_{nn}}x_{n-1}^{(r+1)} \end{aligned} \right\}$$

Thus, the general procedure can be written in the following compact form

$$x_i^{(r+1)} = \frac{b_i}{a_{ii}} - \sum_{j=1}^i \frac{a_{ij}}{a_{ii}} x_j^{(r+1)} - \sum_{j=i+1}^n \frac{a_{ij}}{a_{ii}} x_j^{(r)} \quad \text{for all } i = 1, 2, \dots, n \text{ and } r = 1, 2, \dots$$

To describe system in the first equation, we substitute the r -th approximation into the right-hand side and denote the result by $x_1^{(r+1)}$. In the second equation, we substitute $(x_1^{(r+1)}, x_3^{(r)}, \dots, x_n^{(r)})$ and denote the result by $x_2^{(r+1)}$. In the third equation, we substitute $(x_1^{(r+1)}, x_2^{(r+1)}, x_4^{(r)}, \dots, x_n^{(r)})$ and denote the result by $x_3^{(r+1)}$, and so on. This process is continued till we arrive at the desired result. For illustration, we consider the following example :

Note

The difference between jacobi's method and gauss Seidel method is that in jacobi's method the approximation calculated are used in the next iteration for next approximation but in Gauss-seidel method the new approximation calculated is instantly replaced by the previous one.

Example

Find the solution of the following system of equations using Gauss-Seidel method and perform the first five iterations:

$$\begin{aligned} 4x_1 - x_2 - x_3 &= 2 \\ -x_1 + 4x_2 - x_4 &= 2 \\ -x_1 + 4x_3 - x_4 &= 1 \\ -x_2 - x_3 + 4x_4 &= 1 \end{aligned}$$

Solution

The given system of equations can be rewritten as

$$\left. \begin{aligned} x_1 &= 0.5 + 0.25x_2 + 0.25x_3 \\ x_2 &= 0.5 + 0.25x_1 + 0.25x_4 \\ x_3 &= 0.25 + 0.25x_1 + 0.25x_4 \\ x_4 &= 0.25 + 0.25x_2 + 0.25x_3 \end{aligned} \right\}$$

Taking $x_2 = x_3 = x_4 = 0$ on the right-hand side of the first equation of the system, we get $x_1^{(1)} = 0.5$. Taking $x_3 = x_4 = 0$ and the current value of x_1 , we get from the 2nd equation of the system

$$x_2^{(1)} = 0.5 + (0.25)(0.5) + 0 = 0.625$$

Further, we take $x_4 = 0$ and the current value of x_1 we obtain from the third equation of the system

$$\begin{aligned} x_3^{(1)} &= 0.25 + (0.25)(0.5) + 0 \\ &= 0.375 \end{aligned}$$

Now, using the current values of x_2 and x_3 the fourth equation of system gives

$$x_4^{(1)} = 0.25 + (0.25)(0.625) \\ + (0.25)(0.375) = 0.5$$

The Gauss-Seidel iterations for the given set of equations can be written as

$$x_1^{(r+1)} = 0.5 + 0.25x_2^{(r)} + 0.25x_3^{(r)}$$

$$x_2^{(r+1)} = 0.5 + 0.25x_1^{(r+1)} + 0.25x_4^{(r)}$$

$$x_3^{(r+1)} = 0.25 + 0.25x_1^{(r+1)} + 0.25x_4^{(r)}$$

$$x_4^{(r+1)} = 0.25 + 0.25x_2^{(r+1)} + 0.25x_3^{(r+1)}$$

Now, by Gauss-Seidel procedure, the 2nd and subsequent approximations can be obtained and the sequence of the first five approximations are tabulated as below:

Iteration number r	Variables			
	x1	x2	x3	x4
1	0.5	0.625	0.375	0.5
2	0.75	0.8125	0.5625	0.59375
3	0.84375	0.85938	0.60938	0.61719
4	0.86719	0.87110	0.62110	0.62305
5	0.87305	0.87402	0.62402	0.62451

Example

Solve the system by Gauss-Seidel iterative method

$$8x - 3y + 2z = 20$$

$$4x + 11y - z = 33$$

$$6x + 3y + 12z = 35$$

(Perform only four iterations)

Solution

Consider the given system as

$$8x - 3y + 2z = 20$$

$$4x + 11y - z = 33$$

$$6x + 3y + 12z = 35$$

the system is diagonally dominant

$$x = \frac{1}{8}[20 + 3y - 2z]$$

$$y = \frac{1}{11}[33 - 4x + z]$$

$$z = \frac{1}{12}[35 - 6x - 3y]$$

we start with an initial approximation $x_0 = y_0 = z_0 = 0$

substituting these

first iteration

$$x_1 = \frac{1}{8}[20 + 3(0) - 2(0)] = 2.5$$

$$y_1 = \frac{1}{11}[33 - 4(2.5) + 0] = 2.0909091$$

$$z_1 = \frac{1}{12}[35 - 6(2.5) - 3(2.0909091)] = 1.1439394$$

Second iteration

$$x_2 = \frac{1}{8}[20 + 3y_1 - z_1] = \frac{1}{8}[20 + 3(2.0909091) - 2(1.1439394)] = 2.9981061$$

$$y_2 = \frac{1}{11}[33 - 4x_2 + z_1] = \frac{1}{11}[33 - 4(2.9981061) + 1.1439394] = 2.0137741$$

$$z_2 = \frac{1}{12}[35 - 6x_2 - 3y_2] = \frac{1}{12}[35 - 6(2.9981061) - 3(2.0137741)] = 0.9141701$$

third iteration

$$x_3 = \frac{1}{8}[20 + 3(2.0137741) - 2(0.9141701)] = 3.0266228$$

$$y_3 = \frac{1}{11}[33 - 4(3.0266228) + 0.9141701] = 1.9825163$$

$$z_3 = \frac{1}{12}[35 - 6(3.0266228) - 3(1.9825163)] = 0.9077262$$

fourth iteration

$$x_4 = \frac{1}{8} [20 + 3(1.9825163) - 2(0.9077262)] = 3.0165121$$

$$y_4 = \frac{1}{11} [33 - 4(3.0165121) + 0.9077262] = 1.9856071$$

$$z_4 = \frac{1}{12} [35 - 6(3.0165121) - 3(1.9856071)] = 0.8319128$$

Example

Solve the system by using Gauss-seidel iteration method

$$28x + 4y - z = 32$$

$$x + 3y + 10z = 24$$

$$2x + 17y + 4z = 35$$

Solution

$$28x + 4y - z = 32$$

$$x + 3y + 10z = 24$$

$$2x + 17y + 4z = 35$$

the given system is diagonally dominant so we will make it diagonally dominant by interchanging the equations

$$28x + 4y - z = 32$$

$$2x + 17y + 4z = 35$$

$$x + 3y + 10z = 24$$

hence we can apply Gauss – Seidel method

from the above equations

$$x = \frac{1}{28} [32 - 4y + z]$$

$$y = \frac{1}{17} [35 - 2x - 4z]$$

$$z = \frac{1}{10} [24 - x - 3y]$$

First approximation

putting $y = z = 0$

$$x_1 = \frac{1}{28}[32] = 1.1428571$$

putting $x = 1.1428571$, $z = 0$

$$y_1 = \frac{1}{17}[35 - 2(1.1428571) - 4(0)] = 1.9243697$$

putting $x = 1.1428571$, $y = 1.9243697$

$$z_1 = \frac{1}{10}[24 - 1.1428571 - 3(1.9243697)] = 1.7084034$$

Second iteration

$$x_2 = \frac{1}{28}[32 - 4(1.9243697) + 1.7084034] = 0.9289615$$

$$y_2 = \frac{1}{17}[35 - 2(0.9289615) - 4(1.7084034)] = 1.5475567$$

$$z_2 = \frac{1}{10}[24 - 0.9289615 - 3(1.5475567)] = 1.8408368$$

third iteration

$$x_3 = \frac{1}{28}[32 - 4(1.5475567) + 1.8428368] = 0.9875932$$

$$y_3 = \frac{1}{17}[35 - 2(0.9875932) - 4(1.8428368)] = 1.5090274$$

$$z_3 = \frac{1}{10}[24 - 0.9875932 - 3(1.5090274)] = 1.8485325$$

fourth iteration

$$x_4 = \frac{1}{28}[32 - 4(1.5090274) + 1.8485325] = 0.9933008$$

$$y_4 = \frac{1}{17}[35 - 2(0.9933008) - 4(1.8428368)] = 1.5070158$$

$$z_4 = \frac{1}{10}[24 - 0.9933008 - 3(1.5070158)] = 1.8485652$$

Example

Using Gauss-Seidel iteration method, solve the system of the equation.

$$10x - 2y - z - w = 3$$

$$-2x + 10y - z - w = 15$$

$$-x - y + 10z - 2w = 27$$

$$-x - y - 2z + 10w = -9$$

(Perform only four iterations)

Solution

$$10x - 2y - z - w = 3$$

$$-2x + 10y - z - w = 15$$

$$-x - y + 10z - 2w = 27$$

$$-x - y - 2z + 10w = -9$$

it is diagonally dominant and we may write equation as

$$x = \frac{1}{10}[3 + 2y + z + w]$$

$$y = \frac{1}{10}[15 + 2x + z + w]$$

$$z = \frac{1}{10}[27 + x + y + 2w]$$

$$w = \frac{1}{10}[-9 + x + y + 2z]$$

first approximation

putting $y = z = w = 0$ on RHS of (1), we get

$$x_1 = 0.3$$

$$y_1 = \frac{1}{10}[15 + 2(0.3)] = 1.56$$

putting $x = 0.3, y = 1.56$ and $w = 0$

$$z_1 = \frac{1}{10}[27 + 0.3 + 1.56] = 2.886$$

putting $x = 0.3, y = 1.56$ and $z = 2.886$

$$w_1 = \frac{1}{10}[-9 + 0.3 + 1.56 + 2(2.886)] = -0.1368$$

second iteration

$$x_2 = \frac{1}{10}[3 + 2(1.56) + 2.886 - 0.1368] = 0.88692$$

$$y_2 = \frac{1}{10}[15 + 2(0.88692) + 2.886 - 0.1368] = 1.952304$$

$$z_2 = \frac{1}{10}[27 + 0.88692 + 1.952304 + 2(-0.1368)] = 2.9565624$$

$$w_2 = \frac{1}{10}[-9 + 0.88692 + 1.952304 + 2(2.9565624)] = -0.0247651$$

third iteration

$$x_3 = \frac{1}{10}[3 + 2(1.952304) + 2.9565624 - 0.0247651] = 0.9836405$$

$$y_3 = \frac{1}{10} [15 + 2(0.9836405) + 2.9565624 - 0.0247651] = 1.9899087$$

$$z_3 = \frac{1}{10} [27 + 0.9836405 + 1.9899087 + 2(-0.0247651)] = 2.9924019$$

$$w_3 = \frac{1}{10} [-9 + 0.983405 + 1.9899087 + 2(2.9924019)] = -0.0041647$$

fourth iteration

$$x_4 = \frac{1}{10} [3 + 2(1.9899087) + 2.9924019 - 0.0041647] = 0.9968054$$

$$y_4 = \frac{1}{10} [15 + 2(0.9968054) + 2.9924019 - 0.0041647] = 1.9981848$$

$$z_4 = \frac{1}{10} [27 + 0.9968054 + 1.9981848 + 2(-0.0041647)] = 2.9986661$$

$$w_4 = \frac{1}{10} [-9 + 0.9968054 + 1.9981848 + 2(2.9986661)] = -0.0007677$$

Note

When to stop the iterative processes, we stop the iterative process when we get the required accuracy means if you are asked that find the accurate up to four places of decimal then we will simply perform up to that iteration after which we will get the required accuracy. If we calculate the root of the equation and its consecutive values are 1.895326125, 1.916366125, 1.919356325, 1.919326355, 1.919327145, 1.919327128. Here the accuracy up to seven places of decimal is achieved so if you are asked to acquire the accuracy up to six places of decimal then we will stop here. But in the solved examples only some iteration are carried out and accuracy is not considered here.

Solution of Linear System of Equations and Matrix Inversion

Relaxation Method

This is also an iterative method and is due to Southwell. To explain the details, consider again the system of equations

$$\left. \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{array} \right\}$$

Let

$$X^{(p)} = (x_1^{(p)}, x_2^{(p)}, \dots, x_n^{(p)})^T$$

be the solution vector obtained iteratively after p-th iteration. If $R_i^{(p)}$ denotes the residual of the i-th equation of system given above, that is of $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$ defined by

$$R_i^{(p)} = b_i - a_{i1}x_1^{(p)} - a_{i2}x_2^{(p)} - \cdots - a_{in}x_n^{(p)}$$

we can improve the solution vector successively by reducing the largest residual to zero at that iteration. This is the basic idea of relaxation method.

To achieve the fast convergence of the procedure, we take all terms to one side and then reorder the equations so that the largest negative coefficients in the equations appear on the diagonal.

Now, if at any iteration, R_i is the largest residual in magnitude, then we give an increment to x_i ; a_{ii} being the coefficient of x_i

$$dx_i = \frac{R_i}{a_{ii}}$$

In other words, we change x_i to $(x_i + dx_i)$ to relax R_i that is to reduce R_i to zero.

Example

Solve the system of equations

$$6x_1 - 3x_2 + x_3 = 11$$

$$2x_1 + x_2 - 8x_3 = -15$$

$$x_1 - 7x_2 + x_3 = 10$$

by the relaxation method, starting with the vector (0, 0, 0).

Solution

At first, we transfer all the terms to the right-hand side and reorder the equations, so that the largest coefficients in the equations appear on the diagonal.

Thus, we get

$$\left. \begin{aligned} 0 &= 11 - 6x_1 + 3x_2 - x_3 \\ 0 &= 10 - x_1 + 7x_2 - x_3 \\ 0 &= -15 - 2x_1 - x_2 + 8x_3 \end{aligned} \right\}$$

after interchanging the 2nd and 3rd equations.

Starting with the initial solution vector (0, 0, 0), that is taking $x_1 = x_2 = x_3 = 0$,

we find the residuals $R_1 = 11$, $R_2 = 10$, $R_3 = -15$

of which the largest residual in magnitude is R_3 , i.e. the 3rd equation has more error and needs immediate attention for improvement.

Thus, we introduce a change, dx_3 in x_3 which is obtained from the formula

$$dx_3 = -\frac{R_3}{a_{33}} = \frac{15}{8} = 1.875$$

Similarly, we find the new residuals of large magnitude and relax it to zero, and so on. We shall continue this process, until all the residuals are zero or very small.

Iteration	Residuals	Maximum Difference	Variables
-----------	-----------	--------------------	-----------

number	R1	R2	R3	R_i	dx_i	x1	x2	x3
0	11	10	-15	-15	1.875	0	0	0
1	9.125	8.125	0	9.125	1.5288	0	0	1.875
2	0.0478	6.5962	-3.0576	6.5962	-0.9423	1.5288	0	1.875

Iteration	Residuals	Maximum Difference	Variables
-----------	-----------	--------------------	-----------

number	R1	R2	R3		Ü	Ü	x1	x2	x3
0	11	10	-15	-15		15/8 =1.875	0	0	0
1	9.125	8.125	0	9.125		-9.125/(-6) =1.5288	0	0	1.875
2	0.0478	6.5962	-3.0576	6.5962		-6.5962/7 =-0.9423	1.5288	0	1.875
3	-2.8747	0.0001	-2.1153	-2.8747		2.8747/(-6) =-0.4791	1.0497	-0.9423	1.875
4	-0.0031	0.4792	-1.1571	-1.1571		1.1571/8 =0.1446	1.0497	-0.9423	1.875

Iteration		Residuals		Maximum Difference	Variables			
number	R1	R2	R3	$\ddot{U}$	$\ddot{U}$	x1	x2	x3
5	-0.1447	0.3346	0.0003	0.3346	-.3346/7 =-0.0478	1.0497	-0.9423	2.0156
6	0.2881	0.0000	0.0475	0.2881	-.2881/(-6) =0.0480	1.0497	-0.9901	2.0196
7	-0.0001	0.048	0.1435	0.1435	=-0.0179	1.0017	-0.9901	2.0156
8	0.0178	0.0659	0.0003	-	-	1.0017	-0.9901	2.0057

At this stage, we observe that all the residuals $R1$, $R2$ and $R3$ are small enough and therefore we may take the corresponding values of x_i at this iteration as the solution. Hence, the numerical solution is given by

$$x_1 = 1.0017, \quad x_2 = -0.9901, \quad x_3 = 2.0017,$$

The exact solution is

$$x_1 = 1.0, \quad x_2 = -1.0, \quad x_3 = 2.0$$

Example

Solve by relaxation method, the equation

$$10x - 2y - 2z = 6$$

$$-x - 10y - 2z = 7$$

$$-x - y + 10z = 8$$

Solution

The residual r_1, r_2, r_3 are given by

$$r_1 = 6 - 10x + 2y + 2z$$

$$r_2 = 7 + x - 10y + 2z$$

$$r_3 = 8 + x + y - 10z$$

The operation table is as follows

x	y	z	r1	r2	r3	
1	0	0	-10	1	1	L1
0	1	0	2	-10	1	L2
0	0	1	2	2	-10	L3

The relaxation table is as follows

x	y	z	r1	r2	r3	
0	0	0	6	7	8	L4
0	0	1	8	9	-2	L5=L4+L3
0	1	0	10	-1	-1	L6=L5+L2
1	0	0	0	0	0	L7=L6+L1

Explanation

(1) In L4, the largest residual is 8. to reduce it, To reduce it, we give an increment of

$$\frac{8}{c_3} = \frac{8}{10} = 0.8 \cong 1$$

the resulting residuals are obtained by

$$L_4 + (1)L_3, \text{ i.e line } L_5$$

(2) In line L_5 the largest residual is 9

$$\text{Increment} = \frac{9}{b_2} = \frac{9}{10} = 0.9 \cong 1$$

The resulting residuals ($= L_6$) $= L_5 + 1.L_2$

(3) In line L_6 , the largest residual is 10

$$\text{Increment} = \frac{10}{a_1} = \frac{10}{10} \cong 1$$

The resulting residuals ($= L_6$) $= L_5 + 1.L_2$

Exact solution is arrived and it is $x=1, y=1, z=1$

Example

Solve the system by relaxation method, the equations

$$9x - y + 2z = 7$$

$$x + 10y - 2z = 15$$

$$2x - 2y - 13z = -17$$

Solution

The residuals r_1, r_2, r_3 are given by

$$9x - y + 2z = 9$$

$$x + 10y - 2z = 15$$

$$2x - 2y - 13z = -17$$

here

$$r_1 = 9 - 9x + y - 2z$$

$$r_2 = 15 - x - 10y + 2z$$

$$r_3 = -17 - 2x + 2y + 13z$$

Operation table

x	y	z	r1	r2	r3
1	0	0	-9	-1	-2
0	1	0	1	-10	2
0	0	1	-2	2	13

Relaxation table is

x	y	z	r1	r2	r3
0	0	0	9	15	-17
0	0	1	7	17	-4
0	1	0	8	7	-2
0.89	0	0	-0.01	6.11	-3.78
0	0.61	0	0.6	0.01	-2.56
0	0	0.19	0.22	0.39	-0.09
0	0.039	0	0.259	0	-0.012

0.028	0	0	0.007	-0.028	-0.068
0	0	0.00523	-0.00346	-1.01754	-0.00001

Then $x=0.89+0.028=0.918$; $y=1+0.61+0.039=1.694$

And $z=1+0.19+0.00523=1.19523$

Now substituting the values of x, y, z in (1), we get

$r_1=9-9(0.918)+1.649-2(1.19523)=-0.00346$

$r_2=15-0.918-10(1.649)+2(1.19523)=-0.1754$

$r_3=-17-2(0.918)+2(1.649)+13(1.19523)=-0.00001$

Which is agreement with the final residuals.

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Solution of Linear System of Equations and Matrix Inversion

Matrix Inversion

Consider a system of equations in the form

$$[A](X) = (B)$$

One way of writing its solution is in the form

$$(X) = [A]^{-1}(B)$$

Thus, the solution to the system can also be obtained if the inverse of the coefficient matrix $[A]$ is known. That is the product of two square matrices is an identity matrix

$$[A][B] = [I]$$

$$\text{then, } [B] = [A]^{-1}$$

$$\text{and } [A] = [B]^{-1}$$

Every square non-singular matrix will have an inverse.

Gauss elimination and Gauss-Jordan methods are popular among many methods available for finding the inverse of a matrix.

Gaussian Elimination Method

In this method, if A is a given matrix, for which we have to find the inverse; at first, we place an identity matrix, whose order is same as that of A , adjacent to A which we call an augmented matrix.

Then the inverse of A is computed in two stages. In the first stage, A is converted into an upper triangular form, using Gaussian elimination method

In the second stage, the above upper triangular matrix is reduced to an identity matrix by row transformations. All these operations are also performed on the adjacently placed identity matrix.

Finally, when A is transformed into an identity matrix, the adjacent matrix gives the inverse of A .

In order to increase the accuracy of the result, it is essential to employ partial pivoting.

Example

Use the Gaussian elimination method to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

Solution

At first, we place an identity matrix of the same order adjacent to the given matrix. Thus, the augmented matrix can be written as

$$\left[\begin{array}{cccccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array} \right]$$

Stage I (Reduction to upper triangular form): Let R_1 , R_2 and R_3 denote the 1st, 2nd and 3rd rows of a matrix. In the 1st column, 4 is the largest element, thus interchanging R_1 and R_2 to bring the pivot element 4 to the place of a_{11} , we have the augmented matrix in the form

$$\left[\begin{array}{cccccc} 4 & 3 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array} \right]$$

Divide R_1 by 4 to get

$$\left[\begin{array}{cccccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array} \right]$$

Perform $R_2 - R_1 \rightarrow$, which gives

$$\left[\begin{array}{cccccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array} \right]$$

Perform $R_3 - 3R_1 \rightarrow R_3$ in the above equation, which yields

$$\left[\begin{array}{cccccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 \\ 0 & \frac{5}{4} & \frac{11}{4} & 0 & -\frac{3}{4} & 1 \end{array} \right]$$

Now, looking at the second column for the pivot, the max $(\frac{1}{4}, \frac{11}{4})$ is $\frac{11}{4}$. Therefore, we interchange R_2 and R_3 in the last equation and get

$$\left[\begin{array}{cccccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{11}{4} & \frac{1}{15} & 0 & -\frac{1}{4} & 1 \\ 0 & \frac{1}{4} & \frac{1}{5} & 1 & -\frac{1}{4} & 0 \\ \hline & 4 & 4 & & 4 & \end{array} \right]$$

Now, divide R_2 by the pivot $a_{22} = 11/4$, and obtain

$$\left[\begin{array}{cccccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{15} & 0 & -\frac{1}{11} & \frac{4}{11} \\ 0 & \frac{1}{4} & \frac{1}{5} & 1 & -\frac{1}{4} & 0 \\ \hline & 4 & 4 & & 4 & \end{array} \right]$$

Performing $R_3 - (1/4)R_2 \rightarrow R_3$ yields

$$\left[\begin{array}{cccccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{15} & 0 & -\frac{1}{11} & \frac{4}{11} \\ 0 & 0 & \frac{11}{10} & 1 & -\frac{11}{2} & \frac{11}{1} \\ \hline & 11 & 11 & 11 & & \end{array} \right]$$

Finally, we divide R_3 by $(10/11)$, thus getting an upper triangular form

$$\left[\begin{array}{cccccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{15} & 0 & -\frac{1}{11} & \frac{4}{11} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{11}{5} & 10 \\ \hline & & & 10 & 5 & 10 \end{array} \right]$$

Stage II

Reduction to an identity matrix $(1/4)R_3 + R_1$ and $(-15/11)R_3 + R_2$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & \frac{11}{40} & \frac{1}{5} & -\frac{1}{40} \\ 0 & 1 & 0 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{11}{10} & 1 & 1 \\ \hline & & & 10 & 5 & 10 \end{array} \right]$$

Finally, performing $R_1 - (3/4)R_2 \rightarrow R_1$ we obtain

Thus, we have

Example

Solution

We have the argumented system

now if $\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x & x & x \\ 21 & 22 & 23 \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$

4

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -7 \end{bmatrix}$$

$$\begin{aligned} x_{11} &= -3 & x_{21} &= 12 & x_{31} &= -5 \\ x_{21} &= \frac{5}{2} & x_{22} &= \frac{-17}{2} & x_{23} &= \frac{7}{2} \\ x_{31} &= \frac{-1}{2} & x_{22} &= \frac{3}{2} & x_{23} &= -\frac{1}{2} \end{aligned}$$

and the inverse matrix is

$$\frac{1}{2} \begin{bmatrix} -6 & 5 & -1 \\ 24 & -17 & 3 \\ -10 & 7 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by back substitution, the three systems of equation may be written as

Example

Fine the inverse of the matrix $\begin{bmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix}$ using gauss elimination method.

Solution

$$[A/I] = \left[\begin{array}{cccccc|ccc} 4 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right]$$

$$\sim \left[\begin{array}{cccccc|ccc} 4 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & -2 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -9 & 3 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -4 & 2 & 4 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad R_2 - \frac{1}{2}R_1, R_3 - \frac{1}{4}R_1$$

now if $\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$ is the inverse of the given matrix, then the system (1) is

equivalent to three systems

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & 5 & -2 \\ 0 & -9 & 3 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -14 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & 5 & -2 \\ 0 & -9 & 3 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 9 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & 5 & -2 \\ 0 & -9 & 3 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x_{11} = \frac{-4}{3}$$

$$x_{21} = \frac{5}{3}$$

$$x_{31} = \frac{7}{3}$$

$$x_{21} = \frac{2}{7}$$

$$x_{22} = \frac{-2}{-8}$$

$$x_{23} = \frac{-3}{10}$$

$$x_{31} = \frac{3}{3}$$

$$x_{22} = \frac{3}{3}$$

$$x_{23} = \frac{-10}{3}$$

and the inverse matrix is

$$\begin{bmatrix} \frac{4}{5} & 2 & \frac{7}{-8} \\ -\frac{3}{5} & -2 & \frac{3}{-8} \\ \frac{7}{3} & -3 & \frac{-10}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -4 & 6 & 7 \\ 5 & -6 & -8 \\ 7 & -9 & -10 \end{bmatrix}$$

Gauss - Jordan Method

This method is similar to Gaussian elimination method, with the essential difference that the stage I of reducing the given matrix to an upper triangular form is not needed.

However, the given matrix can be directly reduced to an identity matrix using elementary row operations.

Example

Find the inverse of the given matrix by Gauss-Jordan method

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

Solution

Let R_1 , R_2 and R_3 denote the 1st, 2nd and 3rd rows of a matrix. We place the identity matrix adjacent to the given matrix. So the augmented matrix is given by

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array} \right]$$

Performing $R_2 - 4R_1 \rightarrow R_2$, we get

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array} \right]$$

Now, performing $R_3 - 3R_1 \rightarrow R_3$, we obtain

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & 0 & 1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array} \right]$$

Carrying out further operations $R_2 + R_1 \rightarrow R_1$ and $R_3 + 2R_2 \rightarrow R_3$,

we arrive at

$$\begin{bmatrix} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 0 & -10 & -11 & 2 & 1 \end{bmatrix}$$

Now, dividing the third row by -10 , we get

$$\begin{bmatrix} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{bmatrix}$$

Further, we perform $R_1 + 4R_3 \rightarrow R_1$, and $R_2 + 5R_3 \rightarrow R_2$ to get

$$\begin{bmatrix} 1 & 0 & 0 & \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ 0 & -1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{bmatrix}$$

Finally, multiplying R_2 by -1 , we obtain

$$\begin{bmatrix} 1 & 0 & 0 & \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{bmatrix}$$

Hence, we have

$$A^{-1} = \begin{bmatrix} \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{bmatrix}$$

Exercise

Solve all the above examples solved by Gauss elimination by using gauss Jordan method.

Eigen Value Problems

Let $[A]$ be an $n \times n$ square matrix. Suppose, there exists a scalar λ and a vector

$$X = (x_1 \ x_2 \ \dots \ x_n)^T$$

such that

$$[A](X) = \lambda(X)$$

$$\frac{d}{dx}(e^{ax}) = a(e^{ax})$$

$$\frac{d^2}{dx^2}(\sin ax) = -a^2(\sin ax)$$

Then λ is the eigen value and X is the corresponding eigenvector of the matrix $[A]$.

We can also write it as $[A - \lambda I](X) = (O)$

This represents a set of n homogeneous equations possessing non-trivial solution, provided

$$|A - \lambda I| = 0$$

This determinant, on expansion, gives an n -th degree polynomial which is called characteristic polynomial of $[A]$, which has n roots. Corresponding to each root, we can solve these equations in principle, and determine a vector called eigenvector.

Finding the roots of the characteristic equation is laborious. Hence, we look for better methods suitable from the point of view of computation. Depending upon the type of matrix $[A]$ and on what one is looking for, various numerical methods are available.

Power Method and Jacobi's Method

Note!

We shall consider only real and real-symmetric matrices and discuss power and Jacobi's methods

Power Method

To compute the largest eigen value and the corresponding eigenvector of the system

$$[A](X) = \lambda(X)$$

where $[A]$ is a real, symmetric or un-symmetric matrix, the power method is widely used in practice.

Procedure

Step 1: Choose the initial vector such that the largest element is unity.

Step 2: The normalized vector $v^{(0)}$ is pre-multiplied by the matrix $[A]$.

Step 3: The resultant vector is again normalized.

Step 4: This process of iteration is continued and the new normalized vector is repeatedly pre-multiplied by the matrix $[A]$ until the required accuracy is obtained.

At this point, the result looks like

$$u^{(k)} = [A]v^{(k-1)} = q_k v^{(k)}$$

Here, q_k is the desired largest eigen value and $v^{(k)}$ is the corresponding eigenvector.

Example

Find the eigen value of largest modulus, and the associated eigenvector of the matrix by power method

$$[A] = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix}$$

Solution

We choose an initial vector $v^{(0)}$

as $(1,1,1)^T$.

Then, compute first iteration

$$u^{(1)} = [A]v^{(0)} = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 12 \\ 14 \end{pmatrix}$$

Now we normalize the resultant vector to get

$$u^{(1)} = 14 \begin{pmatrix} \frac{1}{2} \\ \frac{6}{7} \\ 1 \end{pmatrix} = q_1 v^{(1)}$$

The second iteration gives,

$$\begin{aligned} u^{(2)} &= [A]v^{(1)} = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{6}{7} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{39}{7} \\ \frac{67}{7} \\ \frac{171}{14} \end{pmatrix} \\ &= 12.2143 \begin{pmatrix} 0.456140 \\ 0.783626 \\ 1.0 \end{pmatrix} = q_2 v^{(2)} \end{aligned}$$

Continuing this procedure, the third and subsequent iterations are given in the following slides

$$u^{(3)} = [A]v^{(2)} = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix} \begin{pmatrix} 0.456140 \\ 0.783626 \\ 1.0 \end{pmatrix}$$

$$= \begin{pmatrix} 5.263158 \\ 9.175438 \\ 11.935672 \end{pmatrix} = 11.935672 \begin{pmatrix} 0.44096 \\ 0.776874 \\ 1.0 \end{pmatrix} = q_3 v^{(3)}$$

$$u^{(4)} = [A]v^{(3)} = \begin{pmatrix} 5.18814 \\ 9.07006 \\ 11.86036 \end{pmatrix}$$

$$= 11.8636 \begin{pmatrix} 0.437435 \\ 0.764737 \\ 1.0 \end{pmatrix} = q_4 v^{(4)}$$

$$u^{(5)} = [A]v^{(4)} = \begin{pmatrix} 5.16908 \\ 9.04395 \\ 11.84178 \end{pmatrix}$$

$$= 11.84178 \begin{pmatrix} 0.436512 \\ 0.763732 \\ 1.0 \end{pmatrix} = q_5 v^{(5)}$$

After rounding-off, the largest eigen value and the corresponding eigenvector as accurate to two decimals are

$$\lambda = 11.84 \quad (X) = \begin{pmatrix} 0.44 \\ 0.76 \\ 1.00 \end{pmatrix}$$

Example

Find the first three iterations of the power method of the given matrix

$$\begin{bmatrix} 7 & 6 & -3 \\ -12 & -20 & 24 \\ -6 & -12 & 16 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 7 & 6 & -3 \\ -12 & -20 & 24 \\ -6 & -12 & 16 \end{bmatrix}$$

we choose finital vector as $v^{(0)} = (1,1,1)^t$

first iteration

$$u^{(1)} = [A]v^{(0)} = \begin{bmatrix} 7 & 6 & -3 \\ -12 & -20 & 24 \\ -6 & -12 & 16 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7+6-3 \\ -12-20+24 \\ -6-12+16 \end{bmatrix} = \begin{bmatrix} 10 \\ -8 \\ -2 \end{bmatrix}$$

$$\text{by diagonali sin g } \begin{bmatrix} 10 \\ -8 \\ -2 \end{bmatrix} \quad u^{(1)} = 10 \begin{bmatrix} 1 \\ -0.8 \\ -0.2 \end{bmatrix} = q_1 v^{(1)}$$

second iteration

$$u^{(2)} = [A]v^{(1)} = \begin{bmatrix} 7 & 6 & -3 \\ -12 & -20 & 24 \\ -6 & -12 & 16 \end{bmatrix} \begin{bmatrix} 1 \\ -0.8 \\ -0.2 \end{bmatrix} = \begin{bmatrix} 7-4.8+0.6 \\ -12+16-4.8 \\ -6+9.6-3.2 \end{bmatrix} = \begin{bmatrix} 2.8 \\ -0.8 \\ 0.4 \end{bmatrix}$$

$$\text{by diagonali sin g } \begin{bmatrix} 2.8 \\ -0.8 \\ 0.4 \end{bmatrix} \quad u^{(1)} = 2.8 \begin{bmatrix} 1 \\ -0.2857 \\ 0.1428 \end{bmatrix} = q_2 v^{(2)}$$

third iteration

$$u^{(3)} = [A]v^{(2)} = \begin{bmatrix} 7 & 6 & -3 \\ -12 & -20 & 24 \\ -6 & -12 & 16 \end{bmatrix} \begin{bmatrix} 1 \\ -0.2857 \\ 0.1428 \end{bmatrix} = \begin{bmatrix} 7-1.7142-0.4284 \\ -12+5.714+3.4272 \\ -6+3.4284+2.2848 \end{bmatrix} = \begin{bmatrix} 4.8574 \\ -2.8588 \\ -0.2868 \end{bmatrix}$$

now daigonal sin g

$$\begin{bmatrix} 4.8574 \\ -2.8588 \\ -0.2868 \end{bmatrix} \quad \text{now normali sin g } 4.8574 \begin{bmatrix} 1 \\ -0.5885 \\ -0.0590 \end{bmatrix}$$

Example

Find the first three iteration of the power method applied on the following matrices

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{use } x^0 = (-1, 2, 1)^t$$

Solution

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix} \text{ USE } x^{(0)} = (-1, 2, 1)^t$$

1st iterations

$$u^{(1)} = [A]x^{(0)} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1-2+0 \\ 2+8-2 \\ 0-2+2 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

now we normalize the resultant vector to get

$$u^{(1)} = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = 8 \begin{bmatrix} \frac{-3}{8} \\ 1 \\ 0 \end{bmatrix} = q_1 x^{(1)}$$

$$u^{(2)} = [A]x^{(1)} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{-3}{8} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-3}{8} - 1 + 0 \\ \frac{6}{8} + 4 + 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1.375 \\ 4.75 \\ -1 \end{bmatrix}$$

$$u^{(2)} = \begin{bmatrix} -1.375 \\ 4.75 \\ -1 \end{bmatrix} = 4.75 \begin{bmatrix} \frac{-0.28947}{1} \\ 1 \\ \frac{-0.2152}{1} \end{bmatrix}$$

$$u^{(3)} = [A]x^{(2)} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{-0.28947}{1} \\ 1 \\ \frac{-0.2152}{1} \end{bmatrix} = \begin{bmatrix} \frac{-1.28947}{4.99998} \\ 4.99998 \\ \frac{-1.42104}{-0.28420} \end{bmatrix} = 4.99998 \begin{bmatrix} \frac{-0.25789}{1} \\ 1 \\ \frac{-0.28420}{1} \end{bmatrix}$$

Exercise

Find the largest eigen value and the corresponding eigen vector by power method after fourth iteration starting with the initial vector $v^{(0)} = (0, 0, 1)^T$

$$[A] = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$$

Let

$$\lambda_1, \lambda_2, \dots, \lambda_n$$

be the distinct eigen values of an $n \times n$ matrix $[A]$, such that $|\lambda_1| > |\lambda_2| > \dots > |\lambda_n|$ and

suppose $v_1, v_2, \dots, v_n$ are the corresponding eigen vectors

Power method is applicable if the above eigen values are real and distinct, and hence, the corresponding eigenvectors are linearly independent.

Then, any eigenvector v in the space spanned by the eigenvectors $v_1, v_2, \dots, v_n$ can be written as their linear combination $v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

Pre-multiplying by A and substituting

$$Av_1 = \lambda_1 v_1, \quad Av_2 = \lambda_2 v_2, \quad \dots \quad Av_n = \lambda_n v_n$$

We get

$$Av = \lambda_1 \left[c_1 v_1 + c_2 \left(\frac{\lambda_2}{\lambda_1} \right) v_2 + \dots + c_n \left(\frac{\lambda_n}{\lambda_1} \right) v_n \right]$$

Again, pre-multiplying by A and simplifying, we obtain

$$A^2 v = \lambda_1^2 \left[c_1 v_1 + c_2 \left(\frac{\lambda_2^2}{\lambda_1^2} \right) v_2 + \dots + c_n \left(\frac{\lambda_n^2}{\lambda_1^2} \right) v_n \right]$$

Similarly, we have

$$A^r v = \lambda_1^r \left[c_1 v_1 + c_2 \left(\frac{\lambda_2^r}{\lambda_1^r} \right) v_2 + \dots + c_n \left(\frac{\lambda_n^r}{\lambda_1^r} \right) v_n \right]$$

and

$$A^{r+1} v = (\lambda_1)^{r+1} \left[c_1 v_1 + c_2 \left(\frac{\lambda_2^{r+1}}{\lambda_1^{r+1}} \right) v_2 + \dots + c_n \left(\frac{\lambda_n^{r+1}}{\lambda_1^{r+1}} \right) v_n \right]$$

Now, the eigen value λ_1

can be computed as the limit of the ratio of the corresponding components of $A^r v$ and $A^{r+1} v$.

That is,

$$\lambda_1 = \frac{\lambda_1^{r+1}}{\lambda_1^r} = \lim_{r \rightarrow \infty} \frac{(A^{r+1}v)_p}{(A^r v)_p}, \quad p = 1, 2, \dots, n$$

Here, the index p stands for the p -th component in the corresponding vector. Sometimes, we may be interested in finding the least eigen value and the corresponding eigenvector.

In that case, we proceed as follows.

We note that $[A](X) = \lambda(X)$.

Pre-multiplying by $[A^{-1}]$, we get

$$[A^{-1}][A](X) = [A^{-1}]\lambda(X) = \lambda[A^{-1}](X)$$

Which can be rewritten as

$$[A^{-1}](X) = \frac{1}{\lambda}(X)$$

which shows that the inverse matrix has a set of eigen values which are the reciprocals of the eigen values of $[A]$.

Thus, for finding the eigen value of the least magnitude of the matrix $[A]$, we have to apply power method to the inverse of $[A]$.

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$$\begin{aligned}
 & \begin{bmatrix} d_{ii} & d_{ij} \\ d_{ji} & d_{jj} \end{bmatrix} \\
 &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\
 &= \begin{bmatrix} a_{ii} \cos^2\theta + a_{jj} \sin^2\theta & (a_{ii} - a_{jj}) \sin\theta \cos\theta + a_{ij} \cos 2\theta \\ (a_{ji} - a_{ij}) \sin\theta \cos\theta + a_{jj} \cos 2\theta & a_{ii} \sin^2\theta + a_{jj} \cos^2\theta - 2a_{ij} \sin\theta \cos\theta \end{bmatrix}
 \end{aligned}$$

Therefore, $d_{ij} = 0$ only if,

$$a_{ij} \cos 2\theta + \frac{a_{jj} - a_{ii}}{2} \sin 2\theta = 0$$

That is if

$$\tan 2\theta = \frac{2a_{ij}}{a_{ii} - a_{jj}}$$

Thus, we choose θ such that the above equation is satisfied, thereby, the pair of off-diagonal elements d_{ij} and d_{ji} reduces to zero. However, though it creates a new pair of zeros, it also introduces non-zero contributions at formerly zero positions.

Also, the above equation gives four values of θ , but to get the least possible rotation, we choose

Example

Find all the eigen values and the corresponding eigen vectors of the matrix by Jacobi's method

$$A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

Solution

The given matrix is real and symmetric. The largest off-diagonal element is found to be $a_{13} = a_{31} = 2$.

Now, we compute

$$\tan 2\theta = \frac{2a_{ij}}{a_{ii} - a_{jj}} = \frac{2a_{13}}{a_{11} - a_{33}} = \frac{4}{0} = \infty$$

This gives, $\theta = \pi/4$

Thus, we construct an orthogonal matrix S_1 as

$$S_1 = \begin{bmatrix} \cos \frac{\pi}{4} & 0 & -\sin \frac{\pi}{4} \\ 0 & 1 & 0 \\ \sin \frac{\pi}{4} & 0 & \cos \frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

The first rotation gives,

$$D = S_1^{-1} A S_1$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

We observe that the elements d_{13} and d_{31} got annihilated. To make sure that calculations are correct up to this step, we see that the sum of the diagonal elements of D_1 is same as the sum of the diagonal elements of the original matrix A .

As a second step, we choose the largest off-diagonal element of D_1 and is found to be

$d_{12} = d_{21} = 2$, and compute

$$\tan 2\theta = \frac{2d_{12}}{d_{11} - d_{22}} = \frac{4}{0} = \infty$$

This again gives $\theta = \pi/4$

Thus, we construct the second rotation matrix as

$$S_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

At the end of the second rotation, we get

$$\begin{aligned} D &= S_2^{-1} S_1 S_2 \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

This turned out to be a diagonal matrix, so we stop the computation. From here, we notice that the eigen values of the given matrix are 5, 1 and -1. The eigenvectors are the column vectors of $S = S_1 S_2$

Therefore

$$\begin{aligned} S &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

Example

Find all the eigen values of the matrix by Jacobi's method.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Solution

Here all the off-diagonal elements are of the same order of magnitude. Therefore, we can choose any one of them. Suppose, we choose a_{12} as the largest element and compute

$$\tan 2\theta = \frac{-1}{0} = \infty$$

Which gives, $\theta = \pi/4$.

Then $\cos\theta = \sin\theta = 1/\sqrt{2}$

and we construct an orthogonal matrix S_1 such that

$$S = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The first rotation gives

$$D = S^{-1}AS$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 3 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 2 \end{bmatrix}$$

Now, we choose $d_{13} = -1/\sqrt{2}$

As the largest element of D_1 and compute

$$\tan 2\theta = \frac{2d_{13}}{d_{11} - d_{33}} = \frac{-\sqrt{2}}{1 - 2}$$

$$\theta = 27^\circ 22' 41'.$$

Now we construct another orthogonal matrix S_2 , such that

$$S_2 = \begin{bmatrix} 0.888 & 0 & -0.459 \\ 0 & 1 & 0 \\ 0.459 & 0 & 0.888 \end{bmatrix}$$

At the end of second rotation, we obtain

$$D = S_2^{-1}D_1S_2 = \begin{bmatrix} 0.634 & -0.325 & 0 \\ 0.325 & 3 & -0.628 \\ 0 & -0.628 & 2.365 \end{bmatrix}$$

Now, the numerically largest off-diagonal element of D_2 is found to be $d_{23} = -0.628$ and compute.

$$\tan 2\theta = \frac{-2 \times 0.628}{3 - 2.365}$$

$$\theta = -31^\circ 35' 24'.$$

Thus, the orthogonal matrix is

$$S_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.852 & 0.524 \\ 0 & -0.524 & 0.852 \end{bmatrix}$$

At the end of third rotation, we get

$$D = S^{-1} D S = \begin{bmatrix} 0.634 & -0.277 & 0 \\ 0.277 & 3.386 & 0 \\ 0 & 0 & 1.979 \end{bmatrix}$$

To reduce D_3 to a diagonal form, some more rotations are required. However, we may take 0.634, 3.386 and 1.979 as eigen values of the given matrix.

Example

Using Jacobi's method, find the eigenvalues and eigenvectors of the following matrix,

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

Solution:

The given matrix is real and symmetric. The largest off – diagonal element is found to be

$$a_{12} = a_{21} = \frac{1}{2}$$

Now we compute

$$\tan 2\theta = \frac{2a_{ij}}{a_{ii} - a_{jj}} = \frac{2a_{12}}{a_{11} - a_{22}} = \frac{2\left(\frac{1}{2}\right)}{1 - \frac{1}{3}} = 3$$

$$\theta = \frac{\tan^{-1}\left(\frac{3}{2}\right)}{2} = 28.155$$

Thus we construct an orthogonal matrix S_1 as

$$S_1 = \begin{bmatrix} \cos 28.155 & -\sin 28.155 & 0 \\ \sin 28.155 & \cos 28.155 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.882 & -0.472 & 0 \\ 0.472 & 0.882 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The first rotation gives, $D = S_1^{-1} A S_1$

$$\begin{aligned} &= \begin{bmatrix} 0.882 & 0.472 & 0 \\ -0.472 & 0.882 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \begin{bmatrix} 0.882 & -0.472 & 0 \\ 0.472 & 0.882 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1.268 & 0.000 & 0.412 \\ 0.000 & 0.066 & 0.063 \\ 0.412 & 0.063 & 0.200 \end{bmatrix} \end{aligned}$$

We see that sum of the diagonal elements of $D_1 = 1.53$

And the sum of the diagonal elements of $A = 1.53$
This means that our question is going right.

As a second step we choose the largest of off-diagonal element of D_1 , which is $d_{13} = d_{31} = 0.412$, and compute

$$\tan 2\theta = \frac{2d_{13}}{d_{11} - d_{33}} = \frac{2(0.412)}{1.268 - 0.200} = 0.772$$

$$\theta = \frac{\tan^{-1}(0.772)}{2} = 18.834$$

Thus we construct an orthogonal matrix S_2 as

$$S_2 = \begin{bmatrix} \cos 18.834 & 0 & -\sin 18.834 \\ 0 & 1 & 0 \\ \sin 18.834 & 0 & \cos 18.834 \end{bmatrix} = \begin{bmatrix} 0.946 & 0 & -0.323 \\ 0 & 1 & 0 \\ 0.323 & 0 & 0.946 \end{bmatrix}$$

Thus the rotation gives,

$$D = S_2^{-1} D_1 S_2$$

$$= \begin{bmatrix} 0.946 & 0 & 0.323 \\ 0 & 1 & 0 \\ -0.323 & 0 & 0.946 \end{bmatrix} \begin{bmatrix} 1.268 & 0.000 & 0.412 \\ 0.000 & 0.066 & 0.063 \\ 0.412 & 0.063 & 0.200 \end{bmatrix} \begin{bmatrix} 0.946 & 0 & -0.323 \\ 0 & 1 & 0 \\ 0.323 & 0 & 0.946 \end{bmatrix}$$

$$= \begin{bmatrix} 1.408 & 0.020 & -0.001 \\ 0.020 & 0.066 & 0.060 \\ -0.001 & 0.060 & 0.059 \end{bmatrix}$$

We again see that sum of the diagonal elements of $D_2 = 1.53$

Also the sum of the diagonal elements of $A = 1.53$

This means that our question is going right.

Hence the eigenvalues are 1.408, .066 and .059 and the corresponding eigenvectors are the columns of S . Where

$$S = S_1 S_2$$

$$= \begin{bmatrix} 0.882 & -0.472 & 0 \\ 0.472 & 0.882 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.946 & 0 & -0.323 \\ 0 & 1 & 0 \\ 0.323 & 0 & 0.946 \end{bmatrix}$$

$$= \begin{bmatrix} .8343 & -.472 & -.2848 \\ .446 & .88 & -.1524 \\ .323 & 0 & .946 \end{bmatrix}$$



Interpolation

Introduction

Finite differences play an important role in numerical techniques, where tabulated values of the functions are available.

For instance, consider a function $y = f(x)$.

As x takes values $x_0, x_1, x_2, \dots, x_n$,

Let the corresponding values of y be $y_0, y_1, y_2, \dots, y_n$.

That is, for given a table of values, $(x_k, y_k), k = 0, 1, 2, \dots, n$;

The process of estimating the value of y , for any intermediate value of x , is called interpolation. The method of computing the value of y , for a given value of x , lying outside the table of values of x is known as extrapolation. If the function $f(x)$ is known, the value of y corresponding to any x can be readily computed to the desired accuracy. For interpolation of a tabulated function, the concept of finite differences is important. The knowledge about various finite difference operators and their symbolic relations are very much needed to establish various interpolation formulae.

Finite Difference Operators

Forward Differences

For a given table of values $(x_k, y_k), k = 0, 1, 2, \dots, n$ with equally spaced abscissas of a function $y = f(x)$, we define the forward difference operator Δ as follows

$$\Delta y_i = y_{i+1} - y_i, \quad i = 0, 1, \dots, (n-1)$$

To be explicit, we write

$$\begin{aligned}\Delta y_0 &= y_1 - y_0 \\ \Delta y_1 &= y_2 - y_1 \\ &\vdots \\ \Delta y_{n-1} &= y_n - y_{n-1}\end{aligned}$$

These differences are called *first differences of the function* y and are denoted by the symbol Δy_i . Here, Δ is called the first difference operator.

Similarly, the differences of the first differences are called second differences, defined by

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0, \quad \Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

Thus, in general

$$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i$$

Here Δ^2 is called the *second difference operator*. Thus, continuing, we can define, r -th difference of y , as

$$\Delta^r y_i = \Delta^{r-1} y_{i+1} - \Delta^{r-1} y_i$$

By defining a difference table as a convenient device for displaying various differences, the above defined differences can be written down systematically by constructing a difference table for values

$(x_k, y_k), k = 0, 1, \dots, 6$

Forward Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_0	y_0	Δy_0					
x_1	y_1	Δy_1	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$		
x_2	y_2	Δy_2	$\Delta^2 y_1$	$\Delta^3 y_1$	$\Delta^4 y_1$	$\Delta^5 y_0$	
x_3	y_3	Δy_3	$\Delta^2 y_2$	$\Delta^3 y_2$	$\Delta^4 y_1$	$\Delta^5 y_1$	$\Delta^6 y_0$
x_4	y_4	Δy_4	$\Delta^2 y_3$	$\Delta^3 y_3$			
x_5	y_5	Δy_5	$\Delta^2 y_4$				
x_6	y_6						

This difference table is called *forward difference* table or *diagonal difference* table. Here, each difference is located in its appropriate column, midway between the elements of the previous column.

Please note that the subscript remains constant along each diagonal of the table. The first term in the table, that is y_0 is called the leading term, while the differences

$\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \dots$ are called leading differences

Example

Construct a forward difference table for the following values of x and y :

x	0.1	0.3	0.5	0.7	0.9	1.1	1.3
y	0.003	0.067	0.148	0.248	0.37	0.518	0.697

Solution

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0.1	0.003					
0.3	0.067	0.064	0.017			
0.5	0.148	0.081	0.019	0.002		
0.7	0.248	0.1	0.022	0.003	0.001	
0.9	0.37	0.122	0.026	0.004	0.001	0
1.1	0.518	0.148	0.031	0.005	0.001	0
1.3	0.697	0.179				

Example

Express $\Delta^2 y_0$ and $\Delta^3 y_0$ in terms of the values of the function y .

Solution:

Noting that each higher order difference is defined in terms of the lower order difference, we have

$$\begin{aligned}\Delta^2 y_0 &= \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) \\ &= y_2 - 2y_1 + y_0\end{aligned}$$

And

$$\begin{aligned}\Delta^3 y_0 &= \Delta^2 y_1 - \Delta^2 y_0 = (\Delta y_2 - \Delta y_1) - (\Delta y_1 - \Delta y_0) \\ &= (y_3 - y_2) - (y_2 - y_1) - (y_1 - y_0) \\ &= y_3 - 3y_2 + 3y_1 - y_0\end{aligned}$$

Hence, we observe that the coefficients of the values of y , in the expansion of $\Delta^2 y_0$, $\Delta^3 y_0$, are binomial coefficients.

Thus, in general, we arrive at the following result: -

$$\Delta^n y_0 = y_n - {}^n C_1 y_{n-1} + {}^n C_2 y_{n-2} - {}^n C_3 y_{n-3} + \cdots + (-1)^n y_0$$

Example

Show that the value of y_n can be expressed in terms of the leading value y_0 and the leading differences

$$\Delta y_0, \Delta^2 y_0, \dots, \Delta^n y_0$$

Solution

The forward difference table will be

$$\left. \begin{aligned} y_1 - y_0 &= \Delta y_0 \quad \text{or} \quad y_1 = y_0 + \Delta y_0 \\ y_2 - y_1 &= \Delta y_1 \quad \text{or} \quad y_2 = y_1 + \Delta y_1 \\ y_3 - y_2 &= \Delta y_2 \quad \text{or} \quad y_3 = y_2 + \Delta y_2 \end{aligned} \right\}$$

Similarly,

$$\left. \begin{aligned} \Delta y_1 - \Delta y_0 &= \Delta^2 y_0 \quad \text{or} \quad \Delta y_1 = \Delta y_0 + \Delta^2 y_0 \\ \Delta y_2 - \Delta y_1 &= \Delta^2 y_1 \quad \text{or} \quad \Delta y_2 = \Delta y_1 + \Delta^2 y_1 \end{aligned} \right\}$$

Similarly, we can also write

$$\left. \begin{aligned} \Delta^2 y_1 - \Delta^2 y_0 &= \Delta^3 y_0 \quad \text{or} \quad \Delta^2 y_1 = \Delta^2 y_0 + \Delta^3 y_0 \\ \Delta^2 y_2 - \Delta^2 y_1 &= \Delta^3 y_1 \quad \text{or} \quad \Delta^2 y_2 = \Delta^2 y_1 + \Delta^3 y_1 \end{aligned} \right\}$$

$$\begin{aligned} \Delta y_2 &= (\Delta y_0 + \Delta^2 y_0) + (\Delta^2 y_0 + \Delta^3 y_0) \\ &= \Delta y_0 + 2\Delta^2 y_0 + \Delta^3 y_0 \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + \Delta y_2 = (y_1 + \Delta y_1) + (\Delta y_1 + \Delta^2 y_1) \\ &= y_0 + 3\Delta y_0 + 3\Delta^2 y_0 + \Delta^3 y_0 \\ &= (1 + \Delta)^3 y_0 \end{aligned}$$

Similarly, we can symbolically write

$$\begin{aligned} y_1 &= (1 + \Delta) y_0, \\ y_2 &= (1 + \Delta)^2 y_0 \\ y_3 &= (1 + \Delta)^3 y_0 \\ &\dots\dots\dots \\ y_n &= (1 + \Delta)^n y_0 \end{aligned}$$

Hence, we obtain

$$y_n = y_0 + {}^nC_1 \Delta y_0 + {}^nC_2 \Delta^2 y_0 + {}^nC_3 \Delta^3 y_0 + \dots + \Delta^n y_0$$

OR

$$y_n = \sum_{i=0}^n {}^nC_i \Delta^i y_0$$

Backward Differences

For a given table of values (x_k, y_k) , $k = 0, 1, 2, \dots, n$ of a function $y = f(x)$ with equally spaced abscissas, the first backward differences are usually expressed in terms of the backward difference operator ∇ as

$$\nabla y_i = y_i - y_{i-1} \quad i = n, (n-1), \dots, 1$$

To be explicit, we write

$$\begin{aligned} \nabla y_1 &= y_1 - y_0 \\ \nabla y_2 &= y_2 - y_1 \\ &\vdots \\ \nabla y_n &= y_n - y_{n-1} \end{aligned}$$

The differences of these differences are called *second differences* and they are denoted by $\nabla^2 y_2, \nabla^2 y_3, \dots, \nabla^2 y_n$

$$\begin{aligned} \nabla^2 y_2 &= \nabla y_2 - \nabla y_1 \\ \nabla^2 y_3 &= \nabla y_3 - \nabla y_2 \\ &\vdots \\ \nabla^2 y_n &= \nabla y_n - \nabla y_{n-1} \end{aligned}$$

That is

Thus, in general, the second backward differences are

$$\nabla^2 y_i = \nabla y_i - \nabla y_{i-1} \quad i = n, (n-1), \dots, 2$$

While the k -th backward differences are given as

$$\nabla^k y_i = \nabla^{k-1} y_i - \nabla^{k-1} y_{i-1} \quad i = n, (n-1), \dots, k$$

These backward differences can be systematically arranged for a table of values (x_k, y_k) , $k = 0, 1, \dots, 6$ shown below.

Backward Difference Table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$	$\nabla^6 y$
x_0	y_0						
x_1	y_1	∇y_1	$\nabla^2 y_2$	$\nabla^3 y_3$	$\nabla^4 y_4$		
x_2	y_2	∇y_2	$\nabla^2 y_3$	$\nabla^3 y_4$	$\nabla^4 y_5$	$\nabla^5 y_6$	
x_3	y_3	∇y_3	$\nabla^2 y_4$	$\nabla^3 y_5$	$\nabla^4 y_6$	$\nabla^5 y_6$	$\nabla^6 y_6$
x_4	y_4	∇y_4	$\nabla^2 y_5$	$\nabla^3 y_6$			
x_5	y_5	∇y_5	$\nabla^2 y_6$				
x_6	y_6	∇y_6					

From this table, it can be observed that the subscript remains constant along every backward diagonal.

Example

Show that any value of y can be expressed in terms of y_n and its backward differences.

Solution:

From

$$\nabla y_i = y_i - y_{i-1} \quad i = n, (n-1), \dots, 1$$

We get

$$y_{n-1} = y_n - \nabla y_n \quad y_{n-2} = y_{n-1} - \nabla y_{n-1}$$

From

$$\nabla^2 y_i = \nabla y_i - \nabla y_{i-1}, \quad i = n, (n-1), \dots, 2$$

We get

$$\nabla y_{n-1} = \nabla y_n - \nabla^2 y_n$$

From these equations, we obtain

$$y_{n-2} = y_n - 2\nabla y_n + \nabla^2 y_n$$

Similarly, we can show that

$$y_{n-3} = y_n - 3\nabla y_n + 3\nabla^2 y_n - \nabla^3 y_n$$

Symbolically, these results can be rewritten as follows:

$$y_{n-1} = (1 - \nabla) y_n$$

$$y_{n-2} = (1 - \nabla)^2 y_n$$

$$y_{n-3} = (1 - \nabla)^3 y_n$$

.....

$$y_{n-r} = (1 - \nabla)^r y_n$$

$$y_{n-r} = y_n - {}^nC_1 \nabla y_n + {}^nC_2 \nabla^2 y_n - \dots + (-1)^r \nabla^r y_n$$

Central Differences

In some applications, central difference notation is found to be more convenient to represent the successive differences of a function. Here, we use the symbol δ to represent central difference operator and the subscript of δy as for any difference as the average of the subscripts

$$\delta y_{1/2} = y_1 - y_0, \quad \delta y_{3/2} = y_2 - y_1,$$

In General,

$$\delta y_i = y_{i+(1/2)} - y_{i-(1/2)}$$

Higher order differences are defined as follows:

$$\delta^2 y_i = \delta y_{i+(1/2)} - \delta y_{i-(1/2)}$$

$$\delta^n y_i = \delta^{n-1} y_{i+(1/2)} - \delta^{n-1} y_{i-(1/2)}$$

These central differences can be systematically arranged as indicated in the Table

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$	$\delta^6 y$
x_0	y_0						
x_1	y_1	$\delta y_{1/2}$	$\delta^2 y_1$	$\delta^3 y_{3/2}$	$\delta^4 y_2$		
x_2	y_2	$\delta y_{3/2}$	$\delta^2 y_2$	$\delta^3 y_{5/2}$	$\delta^4 y_3$	$\delta^5 y_{5/2}$	
x_3	y_3	$\delta y_{5/2}$	$\delta^2 y_3$	$\delta^3 y_{7/2}$	$\delta^4 y_4$	$\delta^5 y_{7/2}$	$\delta^6 y_3$
x_4	y_4	$\delta y_{7/2}$	$\delta^2 y_4$	$\delta^3 y_{9/2}$			
x_5	y_5	$\delta y_{9/2}$	$\delta^2 y_5$				
x_6	y_6	$\delta y_{11/2}$					

Thus, we observe that all the odd differences have a fractional suffix and all the even differences with the same subscript lie horizontally.

The following alternative notation may also be adopted to introduce finite difference operators. Let $y = f(x)$ be a functional relation between x and y , which is also denoted by y_x .

Suppose, we are given consecutive values of x differing by h say $x, x+h, x+2h, x+3h$, etc. The corresponding values of y are $y_x, y_{x+h}, y_{x+2h}, y_{x+3h}$,

As before, we can form the differences of these values.

Thus

$$\Delta y_x = y_{x+h} - y_x = f(x+h) - f(x)$$

$$\Delta^2 y_x = \Delta y_{x+h} - \Delta y_x$$

Similarly,

$$\nabla y_x = y_x - y_{x-h} = f(x) - f(x-h)$$

$$\delta y_x = y_{x+(h/2)} - y_{x-(h/2)} = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

To be explicit, we write

$$\Delta y_0 = y_1 - y_0$$

$$\Delta y_1 = y_2 - y_1$$

$$\vdots \quad \vdots \quad \vdots$$

$$\Delta y_{n-1} = y_n - y_{n-1}$$

$$\nabla y_i = y_i - y_{i-1} \quad i = n, (n-1), \dots, 1$$

OR

$$\nabla y_1 = y_1 - y_0$$

$$\nabla y_2 = y_2 - y_1$$

$$\vdots \quad \vdots \quad \vdots$$

$$\nabla y_n = y_n - y_{n-1}$$

$$\delta y_{1/2} = y_1 - y_0, \quad \delta y_{3/2} = y_2 - y_1,$$

In General,

$$\delta y_i = y_{i+(1/2)} - y_{i-(1/2)}$$

Higher order differences are defined as follows:

$$\delta^2 y_i = \delta y_{i+(1/2)} - \delta y_{i-(1/2)}$$

$$\delta^n y_i = \delta^{n-1} y_{i+(1/2)} - \delta^{n-1} y_{i-(1/2)}$$

Shift operator, E

Let $y = f(x)$ be a function of x , and let x takes the consecutive values $x, x + h, x + 2h$, etc. We then define an operator having the property

$$E f(x) = f(x + h)$$

Thus, when E operates on $f(x)$, the result is the next value of the function. Here, E is called the shift operator. If we apply the operator E twice on $f(x)$, we get

$$\begin{aligned} E^2 f(x) &= E[E f(x)] \\ &= E[f(x + h)] = f(x + 2h) \end{aligned}$$

Thus, in general, if we apply the operator ' E ' n times on $f(x)$, we get

$$E^n f(x) = f(x + nh)$$

OR

$$E^n y_x = y_{x+nh}$$

$$E y_0 = y_1, \quad E^2 y_0 = y_2, \quad E^4 y_0 = y_4, \quad \dots, \quad E^2 y_{-1} = y_{-1}$$

The inverse operator E^{-1} is defined as

$$E^{-1} f(x) = f(x - h)$$

Similarly

$$E^{-n} f(x) = f(x - nh)$$

Average Operator, μ

it is defined as

$$\begin{aligned} \mu f(x) &= \frac{1}{2} \left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right) \right] \\ &= \frac{1}{2} [y_{x+(h/2)} + y_{x-(h/2)}] \end{aligned}$$

Differential Operator, D

it is defined as

$$\left. \begin{aligned} Df(x) &= \frac{d}{dx} f(x) = f'(x) \\ D^2 f(x) &= \frac{d^2}{dx^2} f(x) = f''(x) \end{aligned} \right\}$$

Important Results Using $\{\Delta, \nabla, \delta, E, \mu\}$

$$\begin{aligned} \Delta y_x &= y_{x+h} - y_x = E y_x - y_x \\ &= (E - 1) y_x \\ \Rightarrow \Delta &= E - 1 \end{aligned}$$

Also

$$\begin{aligned} \nabla y_x &= y_x - y_{x-h} = y_x - E^{-1} y_x \\ &= (1 - E^{-1}) y_x \end{aligned}$$

$$\Rightarrow \nabla = 1 - E^{-1} = \frac{E - 1}{E}$$

And

$$\begin{aligned}\delta y_x &= y_{x+(h/2)} - y_{x-(h/2)} \\ &= E^{1/2} y_x - E^{-1/2} y_x \\ &= (E^{1/2} - E^{-1/2}) y_x\end{aligned}$$

$$\delta = E^{1/2} - E^{-1/2}$$

The definition of μ and E similarly yields

$$\begin{aligned}\mu y_x &= \frac{1}{2} [y_{x+(h/2)} + y_{x-(h/2)}] \\ &= \frac{1}{2} (E^{1/2} + E^{-1/2}) y_x \\ \Rightarrow \mu &= \frac{1}{2} (E^{1/2} + E^{-1/2})\end{aligned}$$

We know that

$$\begin{aligned}E y_x &= y_{x+h} = f(x+h) \\ E y_x &= f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \dots \\ &= f(x) + h D f(x) + \frac{h^2}{2!} D^2 f(x) + \dots \\ &= \left(1 + \frac{hD}{1!} + \frac{h^2 D^2}{2!} + \dots \right) f(x) = e^{hD} y_x\end{aligned}$$

Thus

$$hD = \log E$$

Example:

Prove that

$$\begin{aligned}hD &= \log(1 + \Delta) \\ &= -\log(1 - \nabla) \\ &= \sinh^{-1}(\mu \delta)\end{aligned}$$

Solution:

Using the standard relations we have

$$\begin{aligned}hD &= \log E \\ &= \log(1 + \Delta) \\ &= -\log E^{-1} \\ &= -\log(1 - \nabla)\end{aligned}$$

$$\begin{aligned}
 \mu\delta &= \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) \\
 &= \frac{1}{2}(E - E^{-1}) \\
 &= \frac{1}{2}(e^{hD} - e^{-hD}) \\
 &= \sinh(hD) \\
 \Rightarrow hD &= \sinh^{-1} \mu\delta
 \end{aligned}$$

Example

Prove that

- 1) $1 + \delta^2 \mu^2 = \left(1 + \frac{\delta^2}{2}\right)^2$
- 2) $E^{1/2} = \mu + \frac{\delta}{2}$
- 3) $\Delta = \frac{\delta^2}{2} + \delta \sqrt{1 + (\delta^2/4)}$
- 4) $\mu\delta = \frac{\Delta E^{-1}}{2} + \frac{\Delta}{2}$
- 5) $\mu\delta = \frac{\Delta + \nabla}{2}$

Solution

From the definition, we have:

$$\begin{aligned}
 (1) \quad \mu\delta &= \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) = \frac{1}{2}(E - E^{-1}) \\
 \therefore 1 + \mu^2 \delta^2 &= 1 + \frac{1}{4}(E^2 - 2 + E^{-2}) = \frac{1}{4}(E + E^{-1})^2 \\
 1 + \frac{\delta^2}{2} &= 1 + \frac{1}{2}(E^{1/2} - E^{-1/2})^2 = \frac{1}{2}(E + E^{-1})^2
 \end{aligned}$$

(2)

$$\begin{aligned}
 &\mu + (\delta/2) \\
 &= \frac{1}{2}(E^{1/2} + E^{-1/2} + E^{1/2} - E^{-1/2}) = E^{1/2}
 \end{aligned}$$

(3)

$$\begin{aligned}
 \frac{\delta^2}{2} + \delta \sqrt{1 + (\delta^2/4)} &= \frac{(E^{1/2} - E^{-1/2})^2}{2} + \frac{(E^{1/2} - E^{-1/2}) \sqrt{1 + \frac{1}{4}(E^{1/2} - E^{-1/2})^2}}{1} \\
 &= \frac{E - 2 + E^{-1}}{2} + \frac{1}{2}(E^{1/2} - E^{-1/2})(E^{1/2} + E^{-1/2}) \\
 &= \frac{E - 2 + E^{-1}}{2} + \frac{E - E^{-1}}{2} \\
 &= E - 1 = \Delta
 \end{aligned}$$

(4)

$$\begin{aligned}
 \mu\delta &= \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) = \frac{1}{2}(E - E^{-1}) \\
 &= \frac{1}{2}(1 + \Delta - E^{-1}) = \frac{\Delta}{2} + \frac{1}{2}(1 - E^{-1}) \\
 &= \frac{\Delta}{2} + \frac{1}{2}\left(\frac{E - 1}{E}\right) = \frac{\Delta}{2} + \frac{\Delta}{2E}
 \end{aligned}$$

(5)

$$\begin{aligned}
 \mu\delta &= \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) \\
 &= \frac{1}{2}(E - E^{-1}) \\
 &= \frac{1}{2}(1 + \Delta - 1 + \nabla) = \frac{1}{2}(\Delta + \nabla)
 \end{aligned}$$

Interpolation Newton's Forward Difference Formula

Let $y = f(x)$ be a function which takes values $f(x_0), f(x_0 + h), f(x_0 + 2h), \dots$, corresponding to various equi-spaced values of x with spacing h , say $x_0, x_0 + h, x_0 + 2h, \dots$.

Suppose, we wish to evaluate the function $f(x)$ for a value $x_0 + ph$, where p is any real number, then for any real number p , we have the operator E such that

$$\begin{aligned} E^p f(x) &= f(x + ph). \\ f(x_0 + ph) &= E^p f(x_0) = (1 + \Delta)^p f(x_0) \\ &= \left[1 + p\Delta + \frac{p(p-1)}{2!} \Delta^2 + \frac{p(p-1)(p-2)}{3!} \Delta^3 + \dots \right] f(x_0) \\ f(x_0 + ph) &= f(x_0) + p\Delta f(x_0) \\ &+ \frac{p(p-1)}{2!} \Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{3!} \Delta^3 f(x_0) \\ &+ \dots + \frac{p(p-1)\dots(p-n+1)}{n!} \Delta^n f(x_0) + \text{Error} \end{aligned}$$

This is known as Newton's forward difference formula for interpolation, which gives the value of $f(x_0 + ph)$ in terms of $f(x_0)$ and its leading differences.

This formula is also known as Newton-Gregory forward difference interpolation formula. Here $p = (x - x_0)/h$.

An alternate expression is

$$\begin{aligned} y_x &= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots \\ &+ \frac{p(p-1)(p-n+1)}{n!} \Delta^n y_0 + \text{Error} \end{aligned}$$

If we retain $(r + 1)$ terms, we obtain a polynomial of degree r agreeing with y_x at $x_0, x_1, \dots, x_r$.

This formula is mainly used for interpolating the values of y near the beginning of a set of tabular values and for extrapolating values of y , a short distance backward from y_0 .

Example:

Evaluate $f(15)$, given the following table of values:

x	10	20	30	40	50
$y = f(x)$	46	66	81	93	101

Solution:

The forward differences are tabulated as

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	46				
20	66	20			
30	81	15	-5		
40	93	12	-3	2	
50	101	8	-4	-1	-3

We have Newton's forward difference interpolation formula

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots$$

$$+ \frac{p(p-1)(p-n+1)}{n!}\Delta^n y_0 + \text{Error}$$

Here we have

$$x_0 = 10, \quad y_0 = 46, \quad \Delta y_0 = 20,$$

$$\Delta^2 y_0 = -5, \quad \Delta^3 y_0 = 2, \quad \Delta^4 y_0 = -3$$

Let y_{15} be the value of y when $x = 15$, then

$$p = \frac{x - x_0}{h} = \frac{15 - 10}{10} = 0.5$$

$$f(15) = y_{15} = 46 + (0.5)(20) + \frac{(0.5)(0.5-1)}{2}(-5)$$

$$+ \frac{(0.5)(0.5-1)(0.5-2)}{6}(2) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24}(-3)$$

$$= 46 + 10 + 0.625 + 0.125 + 0.1172$$

$f(15) = 56.8672$ correct to four decimal places.

Example

Find Newton's forward difference, interpolating polynomial for the following data:

x	0.1	0.2	0.3	0.4	0.5
$y = f(x)$	1.4	1.56	1.76	2	2.28

Solution;

The forward difference table to the given data is

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.1	0.14				
0.2	1.56	0.16			
0.3	1.76	0.2	0.04		
0.4	2	0.24	0.04	0	
0.5	2.28	0.28	0.04	0	0

Since, 3rd and 4th leading differences are zero, we have Newton's forward difference interpolating formula as

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0$$

In this problem,

$$x_0 = 0.1, y_0 = 1.40,$$

$$\Delta y_0 = 0.16, \Delta^2 y_0 = 0.04,$$

and

$$p = \frac{x - 0.1}{0.1} = 10x - 1$$

Substituting these values,

$$y = f(x) = 1.40 + (10x - 1)(0.16) + \frac{(10x - 1)(10x - 2)}{2}(0.04)$$

This is the required Newton's interpolating polynomial.

Example

Estimate the missing figure in the following table:

x	1	2	3	4	5
$y = f(x)$	2	5	7	---	32

Solution

Since we are given four entries in the table, the function $y = f(x)$ can be represented by a polynomial of degree three.

$$\Delta^3 f(x) = \text{Constant}$$

$$\text{and } \Delta^4 f(x) = 0, \quad \forall x$$

In particular,

$$\Delta^4 f(x_0) = 0$$

Equivalently,

$$(E - 1)^4 f(x_0) = 0$$

Expanding, we have

$$(E^4 - 4E^3 + 6E^2 - 4E + 1) f(x_0) = 0$$

That is,

$$f(x_4) - 4f(x_3) + 6f(x_2) - 4f(x_1) + f(x_0) = 0$$

Using the values given in the table, we obtain

$$32 - 4f(x_3) + 6 \times 7 - 4 \times 5 + 2 = 0$$

which gives $f(x_3)$, the missing value equal to 14.

Example

Consider the following table of values

x	.2	.3	.4	.5	.6
F(x)	.2304	.2788	.3222	.3617	.3979

Find $f(.36)$ using Newton's Forward Difference Formula.

Solution

x	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.2	0.2304	0.0484	-0.0005	0.0011	-0.0005
0.3	0.2788	0.0434	-0.0039	0.0006	
0.4	0.3222	0.0395	-0.0033		
0.5	0.3617	0.0362			
0.6	0.3979				

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \Delta^4 y_0 + \dots + \frac{p(p-1)(p-2) \dots (p-n+1)}{n!} \Delta^n y_0$$

Where

$$x_0 = 0.2, \quad y_0 = 0.2304, \quad \Delta y_0 = 0.0484, \\ \Delta^2 y_0 = -0.0005, \quad \Delta^3 y_0 = 0.0011, \quad \Delta^4 y_0 = -0.0005 \quad p = \frac{x - x_0}{h} = \frac{0.36 - 0.2}{0.1} = 1.6$$

$$y_x = 0.2304 + 1.6(0.0484) + \frac{1.6(1.6-1)}{2!} (-0.0005) + \frac{1.6(1.6-1)(1.6-2)}{3!} (0.0011) + \frac{1.6(1.6-1)(1.6-2)(1.6-3)}{4!} (-0.0005) \\ = 0.2304 + 0.077441 - 0.0024 + \frac{1.6(1.6-1)(1.6-2)}{6} (0.0011) + \frac{1.6(1.6-1)(1.6-2)(1.6-3)}{24} (-0.0005) \\ = 0.3078 - 0.0024 - 0.00007 - 0.00001 \\ = 0.3053$$

Example

Find a cubic polynomial in x which takes on the values

-3, 3, 11, 27, 57 and 107, when $x = 0, 1, 2, 3, 4$ and 5 respectively.

Solution

Here, the observations are given at equal intervals of unit width.

To determine the required polynomial, we first construct the difference table

Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	-3	6		
1	3	8	2	
2	11	16	8	6
3	27	30	14	6
4	57	30	20	6
5	107	50		

Since the 4th and higher order differences are zero, the required Newton's interpolation formula

$$f(x_0 + ph) = f(x_0) + p\Delta f(x_0) + \frac{p(p-1)}{2}\Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{6}\Delta^3 f(x_0)$$

Here

$$p = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$$

$$\Delta f(x_0) = 6$$

$$\Delta^2 f(x_0) = 2$$

$$\Delta^3 f(x_0) = 6$$

Substituting these values into the formula, we have

$$f(x) = -3 + 6x + \frac{x(x-1)}{2}(2) + \frac{x(x-1)(x-2)}{6}(6)$$

$$f(x) = x^3 - 2x^2 + 7x - 3,$$

The required cubic polynomial.

NEWTON'S BACKWARD DIFFERENCE INTERPOLATION FORMULA

For interpolating the value of the function $y = f(x)$ near the end of table of values, and to extrapolate value of the function a short distance forward from y_n , Newton's backward interpolation formula is used

Derivation

Let $y = f(x)$ be a function which takes on values $f(x_n), f(x_{n-h}), f(x_{n-2h}), \dots, f(x_0)$ corresponding to equispaced values $x_n, x_{n-h}, x_{n-2h}, \dots, x_0$. Suppose, we wish to evaluate the function $f(x)$ at $(x_n + ph)$, where p is any real

number, then we have the shift operator E , such that

$$f(x_n + ph) = E^p f(x)_n = (E^{-1})^{-p} f(x)_n = (1 - \nabla)^{-p} f(x)_n$$

Binomial expansion yields,

$$f(x_n + ph) = \left[1 + p\nabla + \frac{p(p+1)}{2!} \nabla^2 + \frac{p(p+1)(p+2)}{3!} \nabla^3 + \dots + \frac{p(p+1)(p+2)\dots(p+n-1)}{n!} \nabla^n + \text{Error} \right] f(x)_n$$

That is

$$f(x_n + ph) = f(x)_n + p\nabla f(x)_n + \frac{p(p+1)}{2!} \nabla^2 f(x)_n + \frac{p(p+1)(p+2)}{3!} \nabla^3 f(x)_n + \dots + \frac{p(p+1)(p+2)\dots(p+n-1)}{n!} \nabla^n f(x)_n + \text{Error}$$

This formula is known as Newton's backward interpolation formula. This formula is also known as Newton's-Gregory backward difference interpolation formula.

If we retain $(r+1)$ terms, we obtain a polynomial of degree r agreeing with $f(x)$ at x_n ,

$x_{n-1}, \dots, x_{n-r}$. Alternatively, this formula can also be written as

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_n + \dots + \frac{p(p+1)(p+2)\dots(p+n-1)}{n!} \nabla^n y_n + \text{Error}$$

Here $p = \frac{x - x_n}{h}$

Example

For the following table of values, estimate $f(7.5)$.

x	1	2	3	4	5	6	7	8
$y = f(x)$	1	8	27	64	125	216	343	512

Solution

The value to be interpolated is at the end of the table. Hence, it is appropriate to use Newton's backward interpolation formula. Let us first construct the backward difference table for the given data

Difference Table

x	$y = f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	1				
2	8	7			
3	27	19	12		
4	64	37	18	6	
5	125	61	24	6	0
6	216	91	30	6	0
7	343	127	36	6	0
8	512	169	42	6	0

Since the 4th and higher order differences are zero, the required Newton's backward interpolation formula is

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

In this problem,

$$\begin{aligned} p &= \frac{x - x_n}{h} = \frac{7.5 - 8.0}{1} = -0.5 \\ \nabla y_n &= 169, \quad \nabla^2 y_n = 42, \quad \nabla^3 y_n = 6 \\ y_{7.5} &= 512 + (-0.5)(169) + \frac{(-0.5)(0.5)}{2}(42) \\ &\quad + \frac{(-0.5)(0.5)(1.5)}{6}(6) \\ &= 512 - 84.5 - 5.25 - 0.375 \\ &= 421.875 \end{aligned}$$

Example

The sales for the last five years is given in the table below. Estimate the sales for the year 1979

Year	1974	1976	1978	1980	1982
Sales (in lakhs)	40	43	48	52	57

Solution**Newton's backward difference Table**

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1974	40				
1976	43	3			
1978	48	5	2		
1980	52	4	-1	3	
1982	57	5	1	2	5

In this example,

$$p = \frac{1979-1982}{2} = -1.5$$

and

$$\nabla y_n = 5, \quad \nabla^2 y_n = 1,$$

$$\nabla^3 y_n = 2, \quad \nabla^4 y_n = 5$$

Newton's interpolation formula gives

$$\begin{aligned}
 y_{1979} &= 57 + (-1.5)5 + \frac{(-1.5)(-0.5)}{2}(1) + \frac{(-1.5)(-0.5)(0.5)}{6}(2) \\
 &\quad + \frac{(-1.5)(-0.5)(0.5)(1.5)}{24}(5) \\
 &= 57 - 7.5 + 0.375 + 0.125 + 0.1172
 \end{aligned}$$

Therefore,

$$y_{1979} = 50.1172$$

Example

Consider the following table of values

x	1	1.1	1.2	1.3	1.4	1.5
$F(x)$	2	2.1	2.3	2.7	3.5	4.5

Use Newton's Backward Difference Formula to estimate the value of $f(1.45)$.

Solution

x	$y=F(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1	2					
1.1	2.1	0.1				
1.2	2.3	0.2	0.1			
1.3	2.7	0.4	0.2	0.1		
1.4	3.5	0.8	0.4	0.2	0.1	
1.5	4.5	1	0.2	-0.2	-0.4	-0.5

$$p = \frac{x - x_n}{h} = \frac{1.45 - 1.5}{0.1} = -0.5, \quad \nabla y_n = 1, \quad \nabla^2 y_n = .2, \quad \nabla^3 y_n = -.2, \quad \nabla^4 y_n = -.4,$$

$$\nabla^5 y_n = -.5$$

As we know that

$$y_x = y_n + p \nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_n$$

$$+ \frac{p(p+1)(p+2)(p+3)}{4!} \nabla^4 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)}{5!} \nabla^5 y_n$$

$$y_x = 4.5 + (-0.5)(1) + \frac{(-0.5)(-0.5+1)}{2!} (0.2) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!} (-0.2)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{4!} (-0.4) + \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)(-0.5+4)}{5!} (-0.5)$$

$$= 4.5 - 0.5 - 0.025 + 0.0125 + 0.015625 + 0.068359$$

$$= 4.07148$$

LAGRANGE'S INTERPOLATION FORMULA

