Elementary Mathematics(MTH001)

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LECTURE # 1

Introduction

1.Recommended Books:

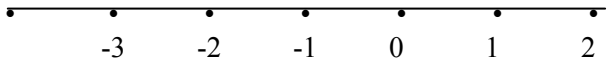
1.Discrete Mathematics with Applications (second edition) by Susanna S. Epp

MAIN TOPICS:

1. Logic
2. Sets & Operations on sets
3. Relations & Their Properties
4. Functions
5. Sequences & Series

Set of Integers:

Set of Real Numbers:



What is Discrete Mathematics?:

Discrete Mathematics concerns processes that consist of a sequence of individual steps.

LOGIC:

Logic is the study of the principles and methods that distinguishes between a valid and an invalid argument.

SIMPLE STATEMENT:

A statement is a declarative sentence that is either true or false but not both.

A statement is also referred to as a **proposition**

Example: 2+2 = 4, It is Sunday today

If a proposition is true, we say that it has a **truth value** of "**true**".

If a proposition is false, its **truth value** is "**false**".

The truth values **"true"** and **"false"** are, respectively, denoted by the letters **T** and **F**.

EXAMPLES:

1.Grass is green.

2.4 + 2 = 6

2.4 + 2 = 7

3.There are four fingers in a hand.

are propositions

Not Propisitions

- Close the door.
- *x* is greater than 2.
- He is very rich

are not propositions.

Rule:

If the sentence is preceded by other sentences that make the pronoun or variable reference clear, then the sentence is a statement.

Example:

x = 1

x > 2

x > 2 is a statement with truth-value FALSE.

Example

Bill Gates is an American

He is very rich

He is very rich is a statement with truth-value TRUE.

UNDERSTANDING STATEMENTS:

1.x + 2 is positive. Not a statement

2.May I come in? Not a statement

3.Logic is interesting. A statement

4.It is hot today. A statement

5.-1 > 0 A statement

6.x + *y* = 12 Not a statement

COMPOUND STATEMENT:
Simple statements could be used to build a compound statement.
LOGICAL CONNECTIVES:
EXAMPLES
1. "3 + 2 = 5" **and** "Lahore is a city in Pakistan"
2. "The grass is green" or " It is hot today"
3. "Discrete Mathematics is **not** difficult to me"
AND, OR, NOT are called LOGICAL CONNECTIVES.
SYMBOLIC REPRESENTATION:
Statements are symbolically represented by letters such as ***p, q, r,...***
EXAMPLES:
p = "Islamabad is the capital of Pakistan"
q = "17 is divisible by 3"

CONNECTIVE	MEANINGS	SYMBOL	CALLED
Negation	not	~	Tilde
Conjunction	and	$\wedge$	Hat
Disjunction	or	$\vee$	Vel
Conditional	if…then…	$\rightarrow$	Arrow
Biconditional	if and only if	$\leftrightarrow$	Double arrow

EXAMPLES:
p = "Islamabad is the capital of Pakistan"
q = "17 is divisible by 3"
$p \wedge q$ = "Islamabad is the capital of Pakistan and 17 is divisible by 3"
$p \vee q$ = "Islamabad is the capital of Pakistan or 17 is divisible by 3"
$\sim p$ = "It is not the case that Islamabad is the capital of Pakistan" or simply "Islamabad is not the capital of Pakistan"
TRANSLATING FROM ENGLISH TO SYMBOLS:
Let p = "It is hot", and q = " It is sunny"

SENTENCE	SYMBOLIC FORM
1.It is **not** hot.	$\sim p$
2.It is hot **and** sunny.	$p \wedge q$
3.It is hot **or** sunny.	$p \vee q$
4.It is **not** hot **but** sunny.	$\sim p \wedge q$
5.It is **neither** hot **nor** sunny.	$\sim p \wedge \sim q$

EXAMPLE:

Let ***h*** = "Zia is healthy"
w = "Zia is wealthy"
s = "Zia is wise"

Translate the compound statements to symbolic form:

1.Zia is healthy and wealthy but not wise. $(h \wedge w) \wedge (\sim s)$

2.Zia is not wealthy but he is healthy and wise. $\sim w \wedge (h \wedge s)$

3.Zia is neither healthy, wealthy nor wise. $\sim h \wedge \sim w \wedge \sim s$

TRANSLATING FROM SYMBOLS TO ENGLISH:

Let m = "Ali is good in Mathematics"
c = "Ali is a Computer Science student"

Translate the following statement forms into plain English:

1.$\sim c$ Ali is not a Computer Science student

2.$c \vee m$ Ali is a Computer Science student or good in Maths.

3.$m \wedge \sim c$ Ali is good in Maths but not a Computer Science student

A convenient method for analyzing a compound statement is to make a truth table for it.

A **truth table** specifies the truth value of a compound proposition for all possible truth values of its constituent propositions.

NEGATION (~):

If ***p*** is a statement variable, then negation of ***p, "not p"***, is denoted as **"~p"**
It has opposite truth value from p i.e.,
if p is true, ~p is false; if p is false, ~p is true.

TRUTH TABLE FOR ~ p

p	*~ p*
T	F
F	T

CONJUNCTION ($\wedge$):

If ***p*** and ***q*** are statements, then the conjunction of ***p*** and ***q*** is ***"p and q",*** denoted as ***"$p \wedge q$".***

It is true when, and only when, both p and q are true. If either p or q is false, or if both are false, $p \wedge q$ is false.

TRUTH TABLE FOR
$p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

DISJUNCTION ($\vee$)
or INCLUSIVE OR

If ***p*** & ***q*** are statements, then the disjunction of ***p*** and ***q*** is ***"p or q",*** denoted as ***"$p \vee q$".*** It is true when at least one of p or q is true and is false only when both p and q are false.

TRUTH TABLE FOR
$p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Note that in the table T is only in that row where both p and q have T and all other values are F. Thus for finding out the truth values for the conjunction of two statements we will only first search out where the both statements are true and write down the T in the corresponding row in the column of $p \wedge q$ and in all other rows we will write F in the column of $p \wedge q$.

DISJUNCTION ($\vee$) or INCLUSIVE OR

If ***p*** and ***q*** are statements, then the disjunction of ***p*** and ***q*** is ***"p or q",*** denoted as ***"$p \vee q$"***

It is true when at least one of p or q is true and is false only when both p and q are false.

Note that in the table F is only in that row where both p and q have F and all other values are T. Thus for finding out the truth values for the disjunction of two statements we will only first search out where the both statements are false and write down the F in the corresponding row in the column of $p \vee q$ and in all other rows we will write T in the column of $p \vee q$.

Remark:

Note that for Conjunction of two statements we find the T in both the statements, But in disjunction we find F in both the statements. In other words we will fill T first in the column of conjunction and F in the column of disjunction.

SUMMARY

1. What is a statement?
2. How a compound statement is formed.
3. Logical connectives (negation, conjunction, disjunction).
4. How to construct a truth table for a statement form.

LECTURE # 2

Truth Tables

Truth Tables for:

1. $\sim p \wedge q$
2. $\sim p \wedge (q \vee \sim r)$
3. $(p \vee q) \wedge \sim (p \wedge q)$

Truth table for the statement form $\sim p \wedge q$

p	q	~p	$\sim p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

Truth table for $\sim p \wedge (q \vee \sim r)$

p	q	r	~ r	$q \vee \sim r$	~ p	$\sim p \wedge (q \vee \sim r)$
T	T	T	F	T	F	F
T	T	F	T	T	F	F
T	F	T	F	F	F	F
T	F	F	T	T	F	F
F	T	T	F	T	T	T
F	T	F	T	T	T	T
F	F	T	F	F	T	F
F	F	F	T	T	T	T

Truth table for $(p \vee q) \wedge \sim (p \wedge q)$

p	q	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$(p \vee q) \wedge \sim (p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Double Negative Property $\sim(\sim p) \equiv p$

p	~p	~(~p)
T	F	T
F	T	F

Example

"It is not true that I am not happy"

Solution:

Let **p** = "I am happy"

then ~ **p** = "I am not happy"

and ~**(~ p)** = "It is not true that I am not happy"

Since ~(~**p**) ≡ **p**

Hence the given statement is equivalent to:

"**I am happy**"

~(p∧q) and ~p ∧ ~q are not logically equivalent

p	q	~p	~q	p∧q	~(p∧q)	~p ∧ ~q
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	F	T	T

Different truth values in row 2 and row 3

DE MORGAN'S LAWS:

1)The negation of an **and** statement is logically equivalent to the **or** statement in which each component is negated.

Symbolically ~(p ∧ q) ≡ ~p ∨ ~q.

2)The negation of an **or** statement is logically equivalent to the **and** statement in which each component is negated.

Symbolically: ~(p ∨ q) ≡ ~p ∧ ~q.

~(p ∨ q) ≡ ~p ∧ ~q

p	q	~p	~q	p ∨ q	~(p ∨ q)	~p ∧ ~q
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Same truth values

Application:

Give negations for each of the following statements:

a.The fan is slow **or** it is very hot.

b.Akram is unfit **and** Saleem is injured.

Solution

a.The fan is **not** slow **and** it is **not** very hot.

b.Akram is **not** unfit **or** Saleem is **not** injured.

INEQUALITIES AND DEMORGAN'S LAWS:

Use DeMorgan's Laws to write the negation of

$$-1 < x \leq 4$$

for some particular real no. x

$-1 < x \leq 4$ means $x > -1$ **and** $x \leq 4$

By DeMorgan's Law, the negation is:

$x > -1$ or $x \leq 4$Which is equivalent to: $x \leq -1$ **or** $x > 4$

EXERCISE:

1. $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
2. Are the statements $(p \wedge q) \vee r$ and $p \wedge (q \vee r)$ logically equivalent?

TAUTOLOGY:

A tautology is a statement form that is always true regardless of the truth values of the statement variables.

A tautology is represented by the symbol "**t**"..

EXAMPLE: The statement form $p \vee \sim p$ is tautology

p	~ p	p ∨ ~ p
T	F	T
F	T	T

$p \vee \sim p \equiv t$

LECTURE # 3

Laws of Logic

APPLYING LAWS OF LOGIC

Using law of logic, simplify the statement form

$p \vee [\sim(\sim p \wedge q)]$

Solution:

$p \vee [\sim(\sim p \wedge q)] \equiv p \vee [\sim(\sim p) \vee (\sim q)]$	DeMorgan's Law
$\equiv p \vee [p \vee (\sim q)]$	Double Negative Law
$\equiv [p \vee p] \vee (\sim q)$	Associative Law for $\vee$
$\equiv p \vee (\sim q)$	Indempotent Law

Which is the simplified statement form.

EXAMPLE Using Laws of Logic, verify the logical equivalence

$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$

$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q)$	DeMorgan's Law
$\equiv (p \vee \sim q) \wedge (p \vee q)$	Double Negative Law
$\equiv p \vee (\sim q \wedge q)$	Distributive Law
$\equiv p \vee c$	Negation Law
$\equiv p$	Identity Law

SIMPLIFYING A STATEMENT:

"You will get an A if you are hardworking and the sun shines, or you are hardworking and it rains." Rephrase the condition more simply.

Solution:

Let p = "You are hardworking'
q = "The sun shines"
r = "It rains" .The condition is then $(p \wedge q) \vee (p \wedge r)$

And using distributive law in reverse,

$(p \wedge q) \vee (p \wedge r) \equiv p \wedge (q \vee r)$

Putting $p \wedge (q \vee r)$ back into English, we can rephrase the given sentence as

"You will get an A if you are hardworking and the sun shines or it rains.

EXERCISE:

Use Logical Equivalence to rewrite each of the following sentences more simply.

1.It is not true that I am tired and you are smart.

{I am not tired or you are not smart.}

2.It is not true that I am tired or you are smart.

{I am not tired and you are not smart.}

3.I forgot my pen or my bag and I forgot my pen or my glasses.

{I forgot my pen or I forgot my bag and glasses.

4.It is raining and I have forgotten my umbrella, or it is raining and I have forgotten my hat.

{It is raining and I have forgotten my umbrella or my hat.}

CONDITIONAL STATEMENTS:

Introduction

Consider the statement:

"**If you earn an A in Math, then I'll buy you a computer**."

This statement is made up of two simpler statements:

p: "You earn an A in Math," and

q: "I will buy you a computer."

The original statement is then saying :

if p is true, then q is true, or, more simply, **if** p, **then** q.

We can also phrase this as p **implies** q, and we write **$p \rightarrow q$**.

CONDITIONAL STATEMENTS OR IMPLICATIONS:

If p and q are statement variables, the conditional of q by p is "If p then q" or "p implies q" and is denoted $p \rightarrow q$.

It is false when p is true and q is false; otherwise it is true. The arrow "$\rightarrow$" is the **conditional** operator, and in $p \rightarrow q$ the statement **p is called the hypothesis (or antecedent)** and q is called the conclusion (or consequent).

TRUTH TABLE:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

PRACTICE WITH CONDITIONAL STATEMENTS:

Determine the truth value of each of the following conditional statements:

1. "If 1 = 1, then 3 = 3." **TRUE**
2. "If 1 = 1, then 2 = 3." **FALSE**
3. "If 1 = 0, then 3 = 3." **TRUE**
4. "If 1 = 2, then 2 = 3." **TRUE**
5. "If 1 = 1,then 1 = 2 and 2 = 3." **FALSE**
6. "If 1 = 3 or 1 = 2 then 3 = 3." **TRUE**

ALTERNATIVE WAYS OF EXPRESSING IMPLICATIONS:

The implication $\mathbf{p \rightarrow q}$ could be expressed in many alternative ways as:

- "if p then q"
- "p implies q"
- "if p, q"
- "p only if q"
- "p is sufficient for q"
- "not p unless q"
- "q follows from p"
- "q if p"
- "q whenever p"
- "q is necessary for p"

EXERCISE:

Write the following statements in the form "if p, then q" in English.

a)Your guarantee is good only if you bought your CD less than 90 days ago.

If your guarantee is good, then you must have bought your CD player less than 90 days ago.

b)To get tenure as a professor, it is sufficient to be world-famous.

If you are world-famous, then you will get tenure as a professor.

c)That you get the job implies that you have the best credentials.

If you get the job, then you have the best credentials.

d)It is necessary to walk 8 miles to get to the top of the Peak.

If you get to the top of the peak, then you must have walked 8 miles.

TRANSLATING ENGLISH SENTENCES TO SYMBOLS:

Let p and q be propositions:

p = "you get an A on the final exam"
q = "you do every exercise in this book"
r = "you get an A in this class"

Write the following propositions using p, q,and r and logical connectives.

1.To get an A in this class it is necessary for you to get an A on the final.

SOLUTION $\mathbf{p \rightarrow r}$

2.You do every exercise in this book; You get an A on the final, implies, you get an A in the class.

SOLUTION $\mathbf{p \wedge q \rightarrow r}$

3. Getting an A on the final and doing every exercise in this book is sufficient For getting an A in this class.

SOLUTION $p \wedge q \rightarrow r$

TRANSLATING SYMBOLIC PROPOSITIONS TO ENGLISH:

Let **p**, **q**, and **r** be the propositions:

p = "you have the flu"

q = "you miss the final exam"

r = "you pass the course"

Express the following propositions as an English sentence.

1. $p \rightarrow q$

If you have flu, then you will miss the final exam.**2.** $\sim q \rightarrow r$

If you don't miss the final exam, you will pass the course.**3.** $\sim p \wedge \sim q \rightarrow r$

If you neither have flu nor miss the final exam, then you will pass the course.

HIERARCHY OF OPERATIONS

FOR LOGICAL CONNECTIVES

- $\sim$ (negation)
- $\wedge$ (conjunction), $\vee$ (disjunction)
- $\rightarrow$ (conditional)

Construct a truth table for the statement form $p \vee \sim q \rightarrow \sim p$

p	q	~ q	~p	p ∨ ~q	**p ∨ ~ q → ~ p**
T	T	F	F	T	F
T	F	T	F	T	F
F	T	F	T	F	T
F	F	T	T	T	T

Construct a truth table for the statement form $(p \rightarrow q) \wedge (\sim p \rightarrow r)$

p	q	r	p→q	~p	~p→r	(p→q)∧(~ p →r)
T	T	T	T	F	T	T
T	T	F	T	F	T	T
T	F	T	F	F	T	F
T	F	F	F	F	T	F
F	T	T	T	T	T	T
F	T	F	T	T	F	F
F	F	T	T	T	T	T
F	F	F	T	T	F	F

LOGICAL EQUIVALENCE INVOLVING IMPLICATION

Use truth table to show $p \rightarrow q \equiv \sim q \rightarrow \sim p$

p	q	~q	~p	p→q	~q → ~p
T	T	F	F	T	T
T	F	T	F	F	F
F	T	F	T	T	T
F	F	T	T	T	T

same truth values

Hence the given two expressions are equivalent.

IMPLICATION LAW

p→q ≡ ~p∨q

p	q	p→q	~p	~p∨q
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

same truth values

NEGATION OF A CONDITIONAL STATEMENT:

Since p→q ≡ ~p∨q therefore

$\sim (p \rightarrow q) \equiv \sim (\sim p \vee q)$

$\equiv \sim (\sim p) \wedge (\sim q)$ by De Morgan's law

$\equiv p \wedge \sim q$ by the Double Negative law

Thus the negation of "**if p then q**" is logically equivalent to "**p and not q**".

Accordingly, the negation of an if-then statement does not start with the word if.

EXAMPLES

Write negations of each of the following statements:

1.If Ali lives in Pakistan then he lives in Lahore.

2.If my car is in the repair shop, then I cannot get to class.

3.If x is prime then x is odd **or** x is 2.

4.If n is divisible by 6, then n is divisible by 2 **and** n is divisible by 3.

SOLUTIONS:

1. Ali lives in Pakistan and he does not live in Lahore.
2. My car is in the repair shop and I can get to class.

3.x is prime but x is not odd **and** x is not 2.

4.n is divisible by 6 but n is not divisible by 2 **or** by 3.

INVERSE OF A CONDITIONAL STATEMENT:

The inverse of the conditional statement $\mathbf{p \rightarrow q}$ is $\mathbf{\sim p \rightarrow \sim q}$

A conditional and its inverse are not equivalent as could be seen from the truth table.

p	q	p→q	~p	~q	~p →~q
T	T	T	F	F	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

different truth values in rows 2 and 3

WRITING INVERSE:

1. ***If today is Friday, then 2 + 3 = 5.***
   If today is not Friday, then $2 + 3 \neq 5$.
2. ***If it snows today, I will ski tomorrow.***
   If it does not snow today I will not ski tomorrow.
3. ***If P is a square, then P is a rectangle.***
   If P is not a square then P is not a rectangle.
4. ***If my car is in the repair shop, then I cannot get to class.***
   If my car is not in the repair shop, then I shall get to the class.

CONVERSE OF A CONDITIONAL STATEMENT:

The converse of the conditional statement $\mathbf{p \rightarrow q}$ is $\mathbf{q \rightarrow p}$

A conditional and its converse are not equivalent.

i.e., $\rightarrow$ is not a commutative operator.

p	q	p→q	q→p
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

WRITING CONVERSE:

1.If today is Friday, then 2 + 3 = 5.
If $2 + 3 = 5$, then today is Friday.

2.If it snows today, I will ski tomorrow.
I will ski tomorrow only if it snows today.

3. ***If P is a square, then P is a rectangle.***
   If P is a rectangle then P is a square.
4. ***If my car is in the repair shop, then I cannot get to class.***
   If I cannot get to the class, then my car is in the repair shop.

CONTRAPOSITIVE OF A CONDITIONAL STATEMENT:

The contrapositive of the conditional statement $\mathbf{p \rightarrow q}$ is~ $\mathbf{q \rightarrow \sim p}$

A conditional and its contrapositive are equivalent. Symbolically, $\mathbf{p \rightarrow q \equiv \sim q \rightarrow \sim p}$

1.If today is Friday, then 2 + 3 = 5.

If $2 + 3 \neq 5$, then today is not Friday.

2.If it snows today, I will ski tomorrow.

I will not ski tomorrow only if it does not snow today.

3. If P is a square, then P is a rectangle.

If P is not a rectangle then P is not a square.

4. If my car is in the repair shop, then I cannot get to class.

If I get to the class, then my car is not in the repair shop.

LECTURE # 4

Biconditional operator

If p and q are statement variables, the biconditional of p and q is "**p if, and only if, q**" and is denoted **p↔q**. *if and only if* abbreviated **iff.**
The double headed arrow " **↔**" is the **biconditional operator.**
TRUTH TABLE FOR
p↔q.

p	q	p ↔q
T	T	T
T	F	F
F	T	F
F	F	T

EXAMPLES:
True or false?
1."**1+1 = 3** if and only if **earth is flat**"
TRUE
2. "**Sky is blue** iff **1 = 0**"
FALSE3. "**Milk is white** iff **birds lay eggs**"
TRUE
4. "**33 is divisible by 4** if and only if **horse has four legs**"
FALSE
5. "**x > 5 iff x^2 > 25**"
FALSE
p↔q ≡ (p→q)∧(q→p)

p	q	p↔q	p→q	q→p	(p→q)∧(q→p)
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

same truth values

REPHRASING BICONDITIONAL:
p↔q is also expressed as:
"p is necessary and sufficient for q"
"if p then q, and conversely"
"p is equivalent to q"
EXERCISE:
Rephrase the following propositions in the form "p if and only if q" in English.
1.If it is hot outside you buy an ice cream cone, and if you buy an ice cream cone it is hot outside.

Sol You buy an ice cream cone if and only if it is hot outside.

2.For you to win the contest it is necessary and sufficient that you have the only winning ticket.

Sol You win the contest if and only if you hold the only winning ticket.

3.If you read the news paper every day, you will be informed and conversely.

Sol You will be informed if and only if you read the news paper every day.**4.It rains if it is a weekend day, and it is a weekend day if it rains.**

Sol It rains if and only if it is a weekend day.

5.The train runs late on exactly those days when I take it.

Sol The train runs late if and only if it is a day I take the train.

6.This number is divisible by 6 precisely when it is divisible by both 2 and 3.

Sol This number is divisible by 6 if and only if it is divisible by both 2 and 3.

TRUTH TABLE FOR

$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$

p	q	$p \rightarrow q$	$\sim q$	$\sim p$	$\sim q \rightarrow \sim p$	$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

TRUTH TABLE FOR

$(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$

p	q	r	$p \leftrightarrow q$	$r \leftrightarrow q$	$(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	F	T
T	F	F	F	T	F
F	T	T	F	T	F
F	T	F	F	F	T
F	F	T	T	F	F
F	F	F	T	T	T

TRUTH TABLE FOR

$p \wedge \sim r \leftrightarrow q \vee r$

Here $p \wedge \sim r \leftrightarrow q \vee r$ means $(p \wedge (\sim r)) \leftrightarrow (q \vee r)$

p	q	r	~r	p∧~r	q∨r	p ∧ ~r ↔q∨r
T	T	T	F	F	T	F
T	T	F	T	T	T	T
T	F	T	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	T	F	T	F	T	F
F	F	T	F	F	T	F
F	F	F	T	F	F	T

LOGICAL EQUIVALENCE
INVOLVING BICONDITIONAL

Show that **~p↔q** and **p↔~q** are logically equivalent

p	q	~p	~q	~p↔q	p↔~q
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	F	F

same truth values

EXERCISE:

Show that **~(p⊕q)** and **p↔q** are logically equivalent

p	q	p⊕q	~(p⊕q)	p↔q
T	T	F	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

same truth values

LAWS OF LOGIC:

1.Commutative Law: $p \leftrightarrow q \equiv q \leftrightarrow p$

2.Implication Laws: $p \rightarrow q \equiv \sim p \vee q$

$\equiv \sim(p \wedge \sim q)$

3.Exportation Law: $(p \wedge q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$

4.Equivalence: $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

5.Reductio ad absurdum $p \rightarrow q \equiv (p \wedge \sim q) \rightarrow c$

APPLICATION:

Rewrite the statement forms without using the symbols $\rightarrow$ or $\leftrightarrow$

1.$p \wedge \sim q \rightarrow r$ **2. $(p \rightarrow r) \leftrightarrow (q \rightarrow r)$**

SOLUTION

1.$p \wedge \sim q \rightarrow r \equiv (p \wedge \sim q) \rightarrow r$ order of operations

$\equiv \sim(p \wedge \sim q) \vee r$ implication law

2.$(p \rightarrow r) \leftrightarrow (q \rightarrow r) \equiv (\sim p \vee r) \leftrightarrow (\sim q \vee r)$ implication law

$\equiv [(\sim p \vee r) \rightarrow (\sim q \vee r)] \wedge [(\sim q \vee r) \rightarrow (\sim p \vee r)]$

equivalence of biconditional

$\equiv [\sim(\sim p \vee r) \vee (\sim q \vee r)] \wedge [\sim(\sim q \vee r) \vee (\sim p \vee r)]$

implication law

Rewrite the statement form **$\sim p \vee q \rightarrow r \vee \sim q$** to a logically equivalent form that uses only $\sim$ and $\wedge$

SOLUTION

STATEMENT	**REASON**
$\sim p \vee q \rightarrow r \vee \sim q$	Given statement form
$\equiv (\sim p \vee q) \rightarrow (r \vee \sim q)$	Order of operations
$\equiv \sim[(\sim p \vee q) \wedge \sim (r \vee \sim q)]$	Implication law $p \rightarrow q \equiv \sim(p \wedge \sim q)$
$\equiv \sim[(\sim p \wedge \sim q) \wedge (\sim r \wedge q)]$	De Morgan's law

Show that **$\sim(p \rightarrow q) \rightarrow p$** is a tautology without using truth tables.

SOLUTIONSTATEMENT	**REASON**
$\sim(p \rightarrow q) \rightarrow p$	Given statement form
$\equiv \sim[\sim(p \wedge \sim q)] \rightarrow p$	Implication law $p \rightarrow q \equiv \sim(p \wedge \sim q)$
$\equiv (p \wedge \sim q) \rightarrow p$	Double negation law
$\equiv \sim(p \wedge \sim q) \vee p$	Implication law $p \rightarrow q \equiv \sim p \vee q$
$\equiv (\sim p \vee q) \vee p$	De Morgan's law
$\equiv (q \vee \sim p) \vee p$	Commutative law of
$\vee \equiv q \vee (\sim p \vee p)$	Associative law of
$\vee \equiv q \vee t$	Negation law
$\equiv t$	Universal bound law

EXERCISE:

Suppose that p and q are statements so that $p \rightarrow q$ is false. Find the truth values of each of the following:

1.$\sim p \rightarrow q$

2.$p \vee q$

3.$q \leftrightarrow p$

SOLUTION

1.TRUE

2.TRUE

3.FALSE

LECTURE # 5

Arguments

EXAMPLE

An interesting teacher keeps me awake. I stay awake in Discrete Mathematics class. Therefore, my Discrete Mathematics teacher is interesting.

Is the above argument valid?

ARGUMENT:

An **argument** is a list of statements called **premises** (or **assumptions** or **hypotheses**) followed by a statement called the **conclusion.**

P_1 Premise

P_2 Premise

P_3 Premise

. . . .

P_n Premise

_____

□C Conclusion

NOTE The symbol \ read "therefore," is normally placed just before the conclusion.

VALID AND INVALID ARGUMENT:

An argument is **valid** if the conclusion is true when all the premises are true.

Alternatively, an argument is valid if conjunction of its premises imply conclusion.That is $(P_1 \wedge P_2 \wedge P_3 \wedge \ldots \wedge P_n) \rightarrow C$ is a tautology.

An argument is **invalid** if the conclusion is false when all the premises are true.

Alternatively, an argument is invalid if conjunction of its premises does not imply conclusion.

EXAMPLE:

Show that the following argument form is valid:

$p \rightarrow q$

p

$\therefore \ q$

SOLUTION

premises conclusion

p	q	p→q	p	q	
T	T	T	T	T	← critical row
T	F	F	T	F	
F	T	T	F	T	
F	F	T	F	F	

EXAMPLE Show that the following argument form is invalid:

$p \rightarrow q$

q

$\therefore \ p$

SOLUTION

premises conclusion

p	q	p→q	q	p
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	T	T	F	F

critical row

EXERCISE:

Use truth table to determine the argument form

$p \vee q$

$p \rightarrow \sim q$

$p \rightarrow r$

$\therefore \quad r$

is valid or invalid.

premises conclusion

p	q	r	p∨q	p→~q	p→r	r
T	T	T	T	F	T	T
T	T	F	T	F	F	F
T	F	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	T	T	T	T
F	T	F	T	T	T	F
F	F	T	F	T	T	T
F	F	F	F	T	T	F

critical rows

The argument form is invalid

LECTURE # 6

Sets

A well defined collection of {distinct}objects is called a set.

- The objects are called the elements or members of the set.
- Sets are denoted by capital letters A, B, C …, X, Y, Z.
- The elements of a set are represented by lower case letters a, b, c, … , x, y, z.
- If an object x is a member of a set A we write x∈A, which reads "x belongs to A" or "x is in A" or "x is an element of A", otherwise we write x∉A, which reads "x does not belong to A" or "x is not in A" or "x is not an element of A".

TABULAR FORM

Listing all the elements of a set, separated by commas and enclosed within braces or curly brackets{}.

EXAMPLES

In the following examples we write the sets in Tabular Form.
A = {1, 2, 3, 4, 5} is the set of first five **Natural Numbers**.
B = {2, 4, 6, 8, …, 50} is the set of **Even numbers** up to 50.
C = {1, 3, 5, 7, 9 …} is the set of **positive odd numbers.**

NOTE

The symbol "…" is called an ellipsis. It is a short for "and so forth."

DESCRIPTIVE FORM:

Stating in words the elements of a set.

EXAMPLES

Now we will write the same examples which we write in Tabular Form ,in the Descriptive Form.

A = set of first five Natural Numbers.(is the Descriptive Form)
B = set of positive even integers less or equal to fifty.
(is the Descriptive Form)
C = {1, 3, 5, 7, 9, …} (is the Descriptive Form)
C = set of positive odd integers. (is the Descriptive Form)

SET BUILDER FORM:

Writing in symbolic form the common characteristics shared by all the elements of the set.

EXAMPLES:

Now we will write the same examples which we write in Tabular as well as Descriptive Form ,in Set Builder Form .

A = {x ÎN / x<=5} (is the Set Builder Form)
B = {x Î E / 0 < x <=50} (is the Set Builder Form)
C = {x ÎO / 0 < x } (is the Set Builder Form)

SETS OF NUMBERS:

1. **Set of Natural Numbers**
   N = {1, 2, 3, … }
2. **Set of Whole Numbers**
   W = {0, 1, 2, 3, … }
3. **Set of Integers**
   Z = {…, -3, -2, -1, 0, +1, +2, +3, …}
   = {0, ±1, ±2, ±3, …}

{"Z" stands for the first letter of the German word for integer: Zahlen.}

4. **Set of Even Integers**
   E = {0, ± 2, ± 4, ± 6, …}
5. **Set of Odd Integers**
   O = {± 1, ± 3, ± 5, …}
6. **Set of Prime Numbers**
   P = {2, 3, 5, 7, 11, 13, 17, 19, …}
7. **Set of Rational Numbers (or Quotient of Integers)**
   Q = {x | x = ; p, q Z, q ≠ 0}
8. **Set of Irrational Numbers**
   Q = Q = { x | x is not rational}
   For example, √2, √3, π, e, etc.
9. **Set of Real Numbers**
   R = Q ∪ Q
10. **Set of Complex Numbers**
    C = {z | z = x + *i*y; x, y R}

What is rational and irrational numbers with examples? A rational number is the one which can be represented in the form of P/Q where P and Q are integers and Q 0. But an irrational number cannot be written in the form of simple fractions. is an example of a rational number whereas 2 is an irrational number.

SUBSET:

If A & B are two sets, A is called a subset of B, written A ⊆ B, if, and only if, any element of A is also an element of B.

Symbolically:

A ⊆ B ⇔ if x A then x B

REMARK:

1. When A ⊆ B, then B is called a superset of A.
2. When A is not subset of B, then there exist at least one x A such that x ∉ B.
3. Every set is a subset of itself.

EXAMPLES:

Let

A = {1, 3, 5} B = {1, 2, 3, 4, 5}
C = {1, 2, 3, 4} D = {3, 1, 5}

Then

A ⊆ B (Because every element of A is in B)
C ⊆ B (Because every element of C is also an element of B)
A ⊆ D (Because every element of A is also an element of D and also note that every element of D is in A so D ⊆ A)
and A is not subset of C .
(Because there is an element 5 of A which is not in C)

EXAMPLE:

The set of integers "Z" is a subset of the set of Rational Number "Q", since every integer 'n' could be written as:

$$n = \frac{n}{1} \in Q$$

Hence Z ⊆ Q.

PROPER SUBSET

Let A and B be sets. A is a proper subset of B, if, and only if, every element of A is in B but there is at least one element of B that is not in A, and is denoted as A ⊂ B.

EXAMPLE:

Let A = {1, 3, 5} B = {1, 2, 3, 5}
then A ⊂ B (Because there is an element 2 of B which is not in A).

EQUAL SETS:

Two sets A and B are equal if, and only if, every element of A is in B and every element of B is in A and is denoted A = B.

Symbolically:

$$A = B \text{ iff } A \subseteq B \text{ and } B \subseteq A$$

EXAMPLE:

Let A = {1, 2, 3, 6} B = the set of positive divisors of 6

C = {3, 1, 6, 2} D = {1, 2, 2, 3, 6, 6, 6}

Then A, B, C, and D are all equal sets.

NULL SET:

A set which contains no element is called a **null set**, or an **empty set** or a **void set**.

It is denoted by the Greek letter ∅ (phi) or { }.

EXAMPLE

A = {x | x is a person taller than 10 feet} = ∅ (Because there does not exist any human being which is taller then 10 feet)

B = {x | x2 = 4, x is odd} = ∅ (Because we know that there does not exist any odd whose square is 4)

REMARK

∅ is regarded as a subset of every set.

EXERCISE:

Determine whether each of the following statements is true or false.

a. $x \in \{x\}$ **TRUE**

(Because x is the member of the singleton set { x })

a. $\{x\} \subseteq \{x\}$ **TRUE**

(Because Every set is the subset of itself.

Note that every Set has necessarily tow subsets ∅ and the Set itself, these two subset are known as Improper subsets and any other subset is called Proper Subset)

a. $\{x\} \in \{x\}$ **FALSE**

(Because { x} is not the member of {x}) Similarly other

d. $\{x\} \in \{\{x\}\}$ **TRUE**

e. $\emptyset \subseteq \{x\}$ **TRUE**

f. $\emptyset \in \{x\}$ **FALSE**

UNIVERSAL SET:

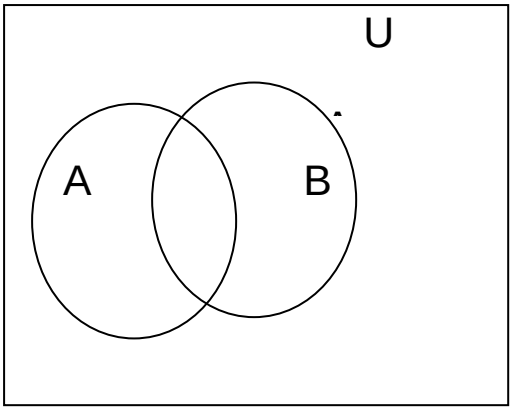
The set of all elements under consideration is called the Universal Set.

The Universal Set is usually denoted by U.

VENN DIAGRAM:

A Venn diagram is a graphical representation of sets by regions in the plane.

The Universal Set is represented by the interior of a rectangle, and the other sets are represented by disks lying within the rectangle.



FINITE AND INFINITE SETS:

A set **S** is said to be **finite** if it contains exactly *m* distinct elements where *m* denotes some non negative integer.

In such case we write |S| = *m* or n(S) = *m*

A set is said to be **infinite** if it is not finite.

EXAMPLES:

1. The set S of letters of English alphabets is finite and |S| = 26
2. The null set ∅ has no elements, is finite and |∅| = 0
3. The set of positive integers {1, 2, 3,…} is infinite.

EXERCISE:

Determine which of the following sets are finite/infinite.

1. A = {month in the year} **FINITE**
2. B = {even integers} **INFINITE**
3. C = {positive integers less than 1} **FINITE**
4. D = {animals living on the earth} **FINITE**
5. E = {lines parallel to x-axis} **INFINITE**
6. F = $\{x \in \mathbf{R} \mid x^{100} + 29x^{50} - 1 = 0\}$ **FINITE**
7. G = {circles through origin} **INFINITE**

MEMBERSHIP TABLE:

A table displaying the membership of elements in sets. To indicate that an element is in a set, a 1 is used; to indicate that an element is not in a set, a 0 is used.

Membership tables can be used to prove set identities.

A	A^c
1	0
0	1

The above table is the Member ship table for Complement of A. now in the above table note that if an element is the member of A then it can`t be the member of A^c thus where in the table we have 1 for A in that row we have 0 in A^c.

LECTURE # 7

Sets Continued

UNION:

Let A and B be subsets of a universal set U. The union of sets A and B is the set of all elements in U that belong to A or to B or to both, and is denoted $A \cup B$.

Symbolically:

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$$

EMAMPLE:

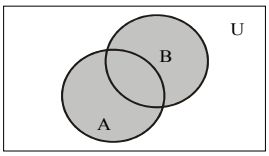
Let U = {a, b, c, d, e, f, g}

A = {a, c, e, g}, B = {d, e, f, g}

Then $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$

$= \{a, c, d, e, f, g\}$

VENN DIAGRAM FOR UNION:



$A \cup B$ is shaded

1+2=3
2+1=3

REMARK:

1. $A \cup B = B \cup A$ that is union is commutative you can prove this very easily only by using definition.

2. $A \subseteq A \cup B$ and $B \subseteq A \cup B$

The above remark of subset is easily seen by the definition of union.

MEMBERSHIP TABLE FOR UNION:

like "OR"

A	B	$A \cup B$
1	1	1
1	0	1
0	1	1
0	0	0

REMARK:

This membership table is similar to the truth table for logical connective, disjunction ($\vee$).

INTERSECTION:

Let A and B subsets of a universal set U. The intersection of sets A and B is the set of all elements in U that belong to both A and B and is denoted $A \cap B$.

Symbolically:

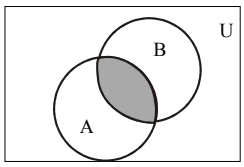
$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$

EXMAPLE:

Let U = {a, b, c, d, e, f, g}

A = {a, c, e, g}, B = {d, e, f, g}

Then $A \cap B = \{e, g\}$



VENN DIAGRAM FOR INTERSECTION: $A \cap B$ is shaded

REMARK:

1. $A \cap B = B \cap A$
2. $A \cap B \subseteq A$ and $A \cap B \subseteq B$
3. If $A \cap B = \phi$, then A & B are called disjoint sets.

MEMBERSHIP TABLE FOR INTERSECTION:

like "AND"

A	B	$A \cap B$
1	1	1
1	0	0
0	1	0
0	0	0

REMARK:

This membership table is similar to the truth table for logical connective, conjunction ($\wedge$).

DIFFERENCE:

Let A and B be subsets of a universal set U. The difference of "A and B" (or relative complement of B in A) is the set of all elements in U that belong to A but not to B, and is denoted A – B or A \ B.

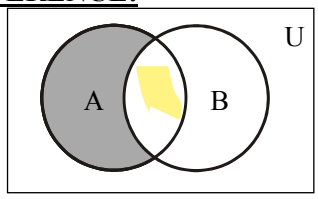
Symbolically:

$$A - B = \{x \in U \mid x \in A \text{ and } x \in B\}$$

EXAMPLE:

Let U = {a, b, c, d, e, f, g}
A = {a, c, e, g}, B = {d, e, f, g}
Then A – B = {a, c}

VENN DIAGRAM FOR SET DIFFERENCE:



A-B is shaded

REMARK:

1. $A - B \neq B - A$ that is Set difference is not commutative.
2. $A - B \subseteq A$
3. A – B, $A \cap B$ and B – A are mutually disjoint sets.

MEMBERSHIP TABLE FOR SET DIFFERENCE:

like "if p than q"

A	B	A – B
1	1	0
1	0	1
0	1	0
0	0	0

REMARK:

The membership table is similar to the truth table for ~ (p →q).

COMPLEMENT:

Let A be a subset of universal set U. The complement of A is the set of all element in U that do not belong to A, and is denoted AN, A or Ac

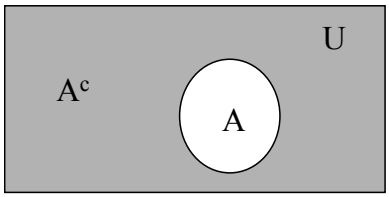
Symbolically:

$A^c = \{x \in U \mid x \notin A\}$

EXAMPLE:

Let U = {a, b, c, d, e, f, g]
A = {a, c, e, g}
Then A^c = {b, d, f}

VENN DIAGRAM FOR COMPLEMENT:



A^c is shaded

REMARK :

1. $A^c = U - A$
2. $A \cap A^c = \phi$
3. $A \cup A^c = U$

MEMBERSHIP TABLE FOR COMPLEMENT: like "negation"

A	A^c
1	0
0	1

REMARK

This membership table is similar to the truth table for logical connective negation (~)

EXERCISE:

Let U = {1, 2, 3, …, 10}, X = {1, 2, 3, 4, 5}
Y = {y | y = 2 x, x ∈X}, Z = {z | z2 – 9 z + 14 = 0}
Enumerate:

(1)X ∩ Y (2) Y ∪ Z (3) X – Z
(4)Y^c (5) $X^c - Z^c$ (6) $(X - Z)^c$

Firstly we enumerate the given sets.
Given

U = {1, 2, 3, …, 10},
X = {1, 2, 3, 4, 5}
Y = {y | y = 2 x, x ∈X} = {2, 4, 6, 8, 10}
Z = {z | z2 – 9 z + 14 = 0} = {2, 7}

(1) X ∩ Y = {1, 2, 3, 4, 5} ∩ {2, 4, 6, 8, 10}
= {2, 4}

(2) Y ∪ Z = {2, 4, 6, 8, 10} ∪ {2, 7}
= {2, 4, 6, 7, 8, 10}

(3) X – Z = {1, 2, 3, 4, 5} – {2, 7}
= {1, 3, 4, 5}

(4) Y^c = U – Y = {1, 2, 3, …, 10} – {2, 4, 6, 8, 10}
= {1, 3, 5, 7, 9

(5) $X^c – Z^c$ = {6, 7, 8, 9, 10} – {1, 3, 4, 5, 6, 8, 9, 10}
= {7}

(6) (X – Z)c = U – (X – Z)
= {1, 2, 3, …, 10} – {1, 3, 4, 5}
= {2, 6, 7, 8, 9, 10}

NOTE (X – Z)c ≠ Xc - Zc

EXERCISE:

Given the following universal set U and its two subsets P and Q, where

U = {x | x ∈ Z,0 ≤ x ≤ 10}
P = {x | x is a prime number}
Q = {x | x2 < 70}

(i) Draw a Venn diagram for the above
(ii) List the elements in Pc ∩ Q

SOLUTION:

First we write the sets in Tabular form.

U = {x | x ∈Z, 0 ≤ x ≤ 10}

Since it is the set of integers that are greater then or equal 0 and less or equal to 10. So we have

U= {0, 1, 2, 3, …, 10}
P = {x | x is a prime number}

It is the set of prime numbers between 0 and 10. Remember Prime numbers are those numbers which have only two distinct divisors.

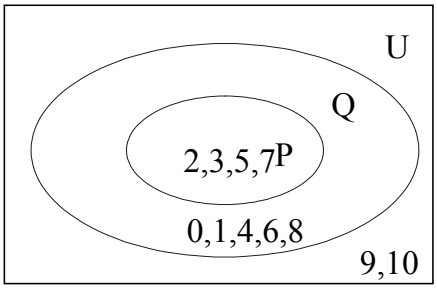
P = {2, 3, 5, 7}
Q = {x | x2 < 70}

The set Q contains the elements between 0 and 10 which has their square less or equal to 70.

Q= {0, 1, 2, 3, 4, 5, 6, 7, 8}

Thus we write the sets in Tabular form.

VENN DIAGRAM:



(i) **P^c ∩ Q = ?**

P^c = U – P = {0, 1, 2, 3, …, 10}- {2, 3, 5, 7}
= {0, 1, 4, 6, 8, 9, 10}

and

$P^c \cap Q = \{0, 1, 4, 6, 8, 9, 10\} \cap \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

$= \{0, 1, 4, 6, 8\}$

EXERCISE:

Let

$U = \{1, 2, 3, 4, 5\}, \quad C = \{1, 3\}$

and A and B are non empty sets. Find A in each of the following:

(i) $A \cup B = U, \quad A \cap B = \phi$ and $B = \{1\}$

(ii) $A \subset B$ and $A \cup B = \{4, 5\}$

(iii) $A \cap B = \{3\}, \quad A \cup B = \{2, 3, 4\}$ and $B \cup C = \{1,2,3\}$

(iv) A and B are disjoint, B and C are disjoint, and the union of A and B is the set $\{1, 2\}$.

(i) $A \cup B = U, \quad A \cap B = \phi$ and $B = \{1\}$

SOLUTION

Since $A \cup B = U = \{1, 2, 3, 4, 5\}$

and $A \cap B = \phi$,

Therefore **$A = B^c = \{1\}^c = \{2, 3, 4, 5\}$**

(i) $A \subset B$ and $A \cup B = \{4, 5\}$ also $C = \{1, 3\}$

SOLUTION

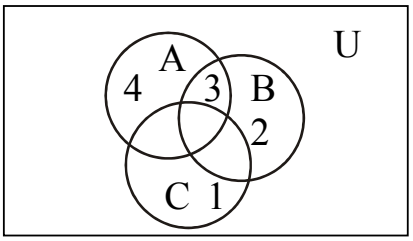
When $A \subset B$, then $A \cup B = B = \{4, 5\}$

Also A being a proper subset of B implies

$A = \{4\}$ or **$A = \{5\}$**

(iii) $A \cap B = \{3\}$, $A \cup B = \{2, 3, 4\}$ and $B \cup C = \{1,2,3\}$
Also $C = \{1, 3\}$

SOLUTION

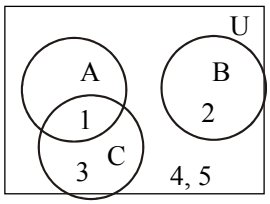


Since we have 3 in the intersection of A and B as well as in C so we place 3 in common part shared by the three sets in the Venn diagram. Now since 1 is in the union of B and C it means that 1 may be in C or may be in B, but 1cannot be in B because if 1 is in the B then it must be in $A \cup B$ but 1 is not there, thus we place 1 in the part of C which is not shared by any other set. Same is the reason for 4 and we place it in the set which is not shared by any other set. Now 2 will be in B, 2 cannot be in A because $A \cap B = \{3\}$, and is not in C.

So $A = \{3, 4\}$ and $B = \{2, 3\}$

(i) $A \cap B = \phi, \quad B \cap C = \phi, \quad A \cup B = \{1, 2\}.$
Also $C = \{1, 3\}$

SOLUTION



A = {1}

EXERCISE:

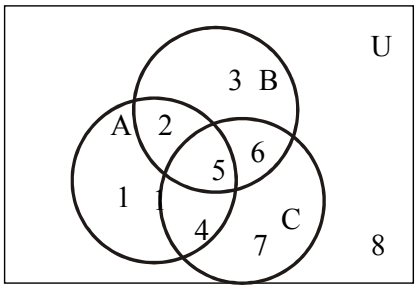
Use a Venn diagram to represent the following:

(i) $(A \cap B) \cap C^c$

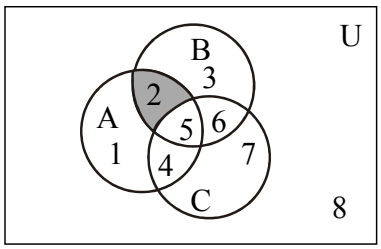
(ii) $A^c \cup (B \cup C)$

(iii) $(A - B) \cap C$

(iv) $(A \cap B^c) \cup C^c$

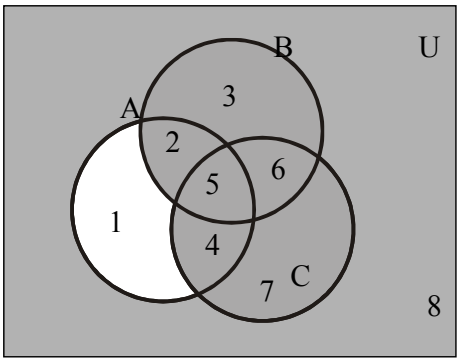


(1) $(A \cap B) \cap C^c$

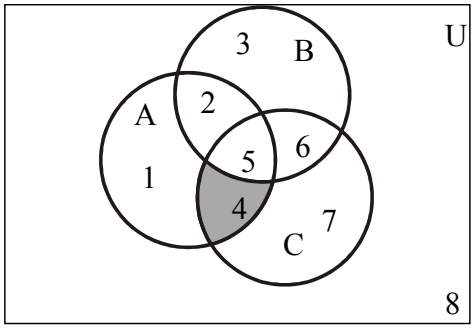


$(A \cap B) \cap C^c$ is shaded

(ii) $A^c \cup (B \cup C)$ is shaded.

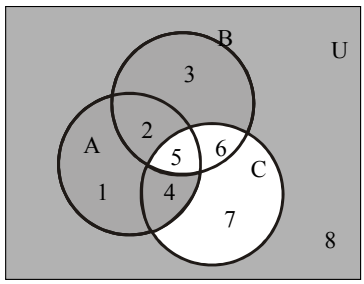


(iii) $(A - B) \cap C$



$(A - B) \cap C$ is shaded

(iii) **$(A \cap B^c) \cup C^c$ is shaded.**



PROVING SET IDENTITIES BY VENN DIAGRAMS:

Prove the following using Venn Diagrams:

(i) $A - (A - B) = A \cap B$

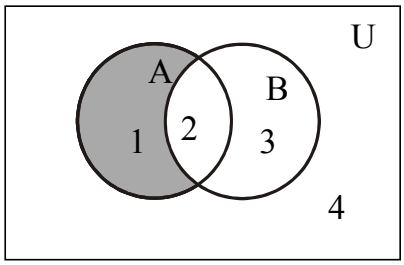
(ii) $(A \cap B)^c = A^c \cup B^c$

(iii) $A - B = A \cap B^c$

SOLUTION (i)

$A - (A - B) = A \cap B$

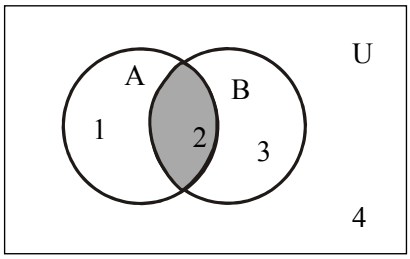
(a)



$A = \{ 1, 2 \}$
$B = \{ 2, 3 \}$
$A - B = \{ 1 \}$

$A - B$ is shaded

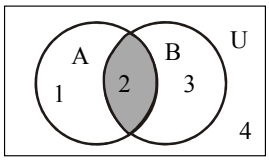
(b)



A = { 1, 2 }
A – B = { 1 }
A – (A – B) = { 2 }

A – (A – B) is shaded

(c)



A ∩ B is shaded

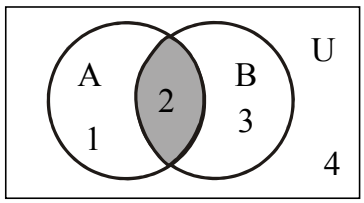
A= { 1, 2 }
B = { 2, 3 }
A ∩ B = {2}

RESULT: A – (A – B) = A ∩ B

SOLUTION (ii)

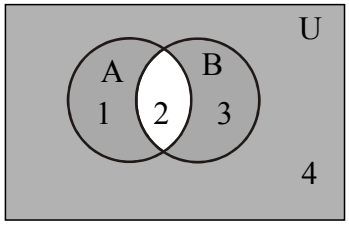
$(A \cap B)^c = A^c \cup B^c$

(a)



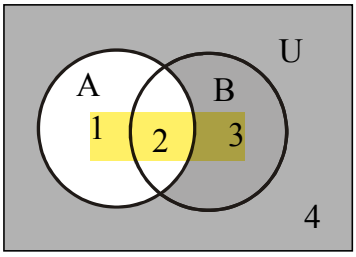
A ∩ B

(b)



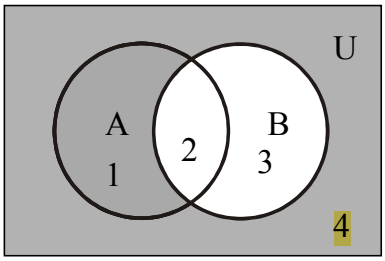
$(A \cap B)^c$

(c)



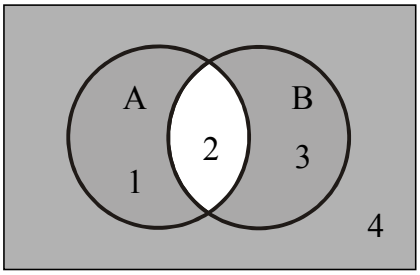
A^c is shaded.

(d)



B^c is shaded.

(e)



$A^c \cup B^c$ is shaded.

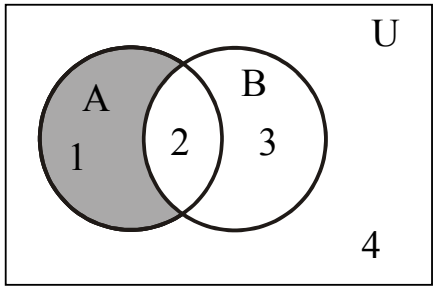
Now diagrams (b) and (e) are same hence

RESULT: $(A \cap B)^c = A^c \cup B^c$

SOLUTION (iii)

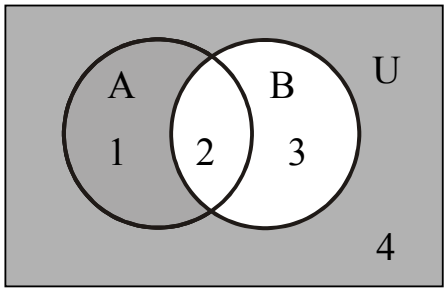
$A - B = A \cap B^c$

(a)



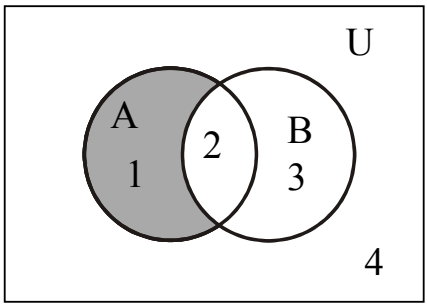
A – B is shaded.

(b)



B^c is shaded.

(c)



$A \cap B^c$ is shaded

From diagrams (a) and (b) we can say

RESULT: $A - B = A \cap B^c$

PROVING SET IDENTITIES BY MEMBERSHIP TABLE:

Prove the following using Membership Table:

(i) $A - (A - B) = A \cap B$

(ii) $(A \cap B)^c = A^c \cup B^c$

(iii) $A - B = A \cap B^c$

SOLUTION (i)

$A - (A - B) = A \cap B$

A	B	A-B	A-(A-B)	A∩B
1	1	0	1	1
1	0	1	0	0
0	1	0	0	0
0	0	0	0	0

Since the last two columns of the above table are same hence the corresponding set expressions are same. That is

$$A - (A - B) = A \cap B$$

SOLUTION (ii)

$$(A \cap B)^c = A^c \cup B^c$$

A	B	A∩B	$(A \cap B)^c$	A^c	B^c	$A^c \cup B^c$
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

Since the fourth and last columns of the above table are same hence the corresponding set expressions are same. That is

$$(A \cap B)^c = A^c \cup B^c$$

SOLUTION (iii)

A	B	A – B	B^c	$A \cap B^c$
1	1	0	0	0
1	0	1	1	1
0	1	0	0	0
0	0	0	1	0

LECTURE # 8

Sets Continued

ORDERED PAIR:

An ordered pair (a, b) consists of two elements "a" and "b" in which "a" is the first element and "b" is the second element.

The ordered pairs (a, b) and (c, d) are equal if, and only if, a= c and b = d.

Note that (a, b) and (b, a) are not equal unless a = b.

EXERCISE:

Find x and y given (2x, x + y) = (6, 2)

SOLUTION:

Two ordered pairs are equal if and only if the corresponding components are equal. Hence, we obtain the equations:

$2x = 6$ ……(1)

and $x + y = 2$ ……..(2)

Solving equation (1) we get $x = 3$ and when substituted in (2) we get $y = -1$.

ORDERED n-TUPLE:

The ordered n-tuple, $(a_1, a_2, \ldots, a_n)$ consists of elements $a_1, a_2, ..a_n$ together with the ordering: first a_1, second a_2, and so forth up to a_n. In particular, an ordered 2-tuple is called an ordered pair, and an ordered 3-tuple is called an ordered triple.

Two ordered n-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are equal if and only if each corresponding pair of their elements is equal, i.e., $a_i = b_j$, for all

i = 1, 2… n.

CARTESIAN PRODUCT OF TWO SETS:

Let A and B be sets. The Cartesian product of A and B, denoted $A \times B$ (read "A cross B") is the set of all ordered pairs (a, b), where a is in A and b is in B.

Symbolically:

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$$

NOTE

If set A has ***m*** elements and set B has ***n*** elements then $A \times B$ has $m \times n$ elements.

EXAMPLE:

Let A = {1, 2}, B = {a, b, c} then

$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2, b), (2, c)\}$

$B \times A = \{(a,1), (a,2), (b, 1), (b, 2), (c, 1), (c, 2)\}$

$A \times A = \{(1, 1), (1,2), (2, 1), (2, 2)\}$

$B \times B = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b),(c, c)\}$

REMARK:

1. $A \times B \neq B \times A$ for non-empty and unequal sets A and B.
2. $A \times \phi = \phi \times A = \phi$
3. $|A \times B| = |A| \times |B|$

CARTESIAN PRODUCT OF MORE THAN TWO SETS:

The Cartesian product of sets $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \ldots \times A_n$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, \text{ for } i=1, 2, \ldots, n\}$$

BINARY RELATION:

Let A and B be sets. A (binary) relation R from A to B is a subset of $A \times B$.

When $(a, b) \in R$, we say a is related to b by R, written a R b.

Otherwise if $(a, b) \notin R$, we write a R b

EXAMPLE:

Let A = {1, 2}, B = {1, 2, 3}

Then $A \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$

Let

$R_1=\{(1,1), (1, 3), (2, 2)\}$

$R_2=\{(1, 2), (2, 1), (2, 2), (2, 3)\}$

$R_3=\{(1, 1)\}$

$R_4= A \times B$

$R_5= \varnothing$

All being subsets of $A \times B$ are relations from A to B.

DOMAIN OF A RELATION:

The domain of a relation R from A to B is the set of all first elements of the ordered pairs which belong to R denoted Dom(R).

Symbolically:

$$Dom(R) = \{a \in A | (a,b) \in R\}$$

RANGE OF A RELATION:

The range of A relation R from A to B is the set of all second elements of the ordered pairs which belong to R denoted Ran(R).

Symbolically:

$$Ran(R) = \{b \in B | (a,b) \in R\}$$

EXERCISE:

Let $A = \{1, 2\}$, $B = \{1, 2, 3\}$,

Define a binary relation R from A to B as follows:

$$R = \{(a, b) \in A \times B \mid a < b\}$$

1/x-5

Then

a. Find the ordered pairs in R.
b. Find the Domain and Range of R.
c. Is 1R3, 2R2?

1/X-2
x=3
1/3-2=
1/4-2=1/2
D=3
R=1
D=4
R=1/2
1/5-2
D=5
R=1/3

SOLUTION:

A=(1,2)
D=1
R=2

Given $A = \{1, 2\}$, $B = \{1, 2, 3\}$,

$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$

a. $R = \{(a, b) \in A \times B \mid a < b\}$

$R = \{(1,2), (1,3), (2,3)\}$

b. $Dom(R) = \{1,2\}$ and $Ran(R) = \{2, 3\}$ subset of A subset of B

a. Since $(1,3) \in R$ so 1R3

But $(2, 2) \notin R$ so 2 is not related with3.

EXAMPLE:

Let A = {eggs, milk, corn} and B = {cows, goats, hens}

Define a relation R from A to B by $(a, b) \in R$ *iff* a is produced by b.

Then R = {(eggs, hens), (milk, cows), (milk, goats)}

Thus, with respect to this relation eggs R hens , milk R cows, etc.

1/2-2
1/0
1/1-2

EXERCISE :

Find all binary relations from {0,1} to {1}

SOLUTION:

Let $A = \{0,1\}$ & $B = \{1\}$

Then $A \times B = \{(0,1), (1,1)\}$

All binary relations from A to B are in fact all subsets of $A \times B$, which are:

$R_1= \varnothing$

$R_2=\{(0,1)\}$

$R_3=\{(1,1)\}$

$R_4=\{(0,1), (1,1)\} = A \times B$

REMARK:

If |A| = m and |B| = n

Then as we know that the number of elements in A × B are m × n. Now as we know that the total number of and the total number of relations from A to B are $2^{m \times n}$.

RELATION ON A SET:

A relation on the set A is a relation from A to A.

In other words, a relation on a set A is a subset of A × A.

EXAMPLE: :

Let A = {1, 2, 3, 4}

Define a relation R on A as

(a,b) ∈ R *iff* a divides b {symbolically written as a | b}

Then R = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

REMARK:

For any set A

1. A × A is known as the universal relation.
2. ∅ is known as the empty relation.

EXERCISE:

Define a binary relation E on the set of the integers Z, as follows:

for all m,n ∈Z, m E n ⇔ m – n is even.

a. Is 0E0? Is 5E2? Is (6,6) ∈E? Is (-1,7) ∈E?
b. Prove that for any even integer n, nE0.

SOLUTION

E = {(m,n) ∈Z ×Z | m – n is even}

a. (i) (0,0) ∈ Z ×Z and 0-0 = 0 is even
Therefore 0E0.

(ii) (5,2) ∈ Z ×Z but 5-2 = 3 is not even
so 5 E 2

(iii) (6,6) ∈ E since 6-6 = 0 is an even integer.

(iv) (-1,7) ∈E since (-1) – 7 = -8 is an even integer.

a. For any even integer, n, we have
n – 0 = n, an even integer
so (n, 0) ∈E or equivalently n E 0

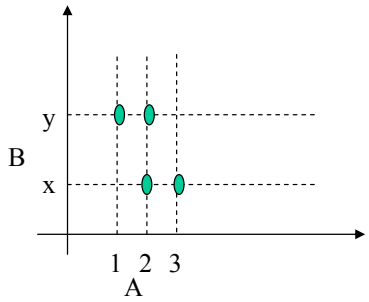
COORDINATE DIAGRAM (GRAPH) OF A RELATION:

Let A = {1, 2, 3} and B = {x, y}

Let R be a relation from A to B defined as

R = {(1, y), (2, x), (2, y), (3, x)}

The relation may be represented in a coordinate diagram as follows:



EXAMPLE:

Draw the graph of the binary relation C from R to R defined as follows:

for all $(x, y) \in R \times R$, $(x, y) \in C \Leftrightarrow x^2 + y^2 = 1$

SOLUTION

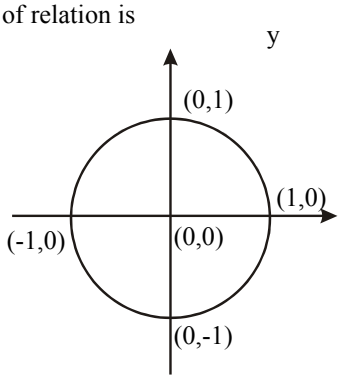
All ordered pairs (x, y) in relation C satisfies the equation $x^2+y^2=1$, which when solved for y gives

Clearly y is real, whenever $-1 \leq x \leq 1$

Similarly x is real, whenever $-1 \leq y \leq 1$

Hence the graph is limited in the range $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$

The graph of relation is



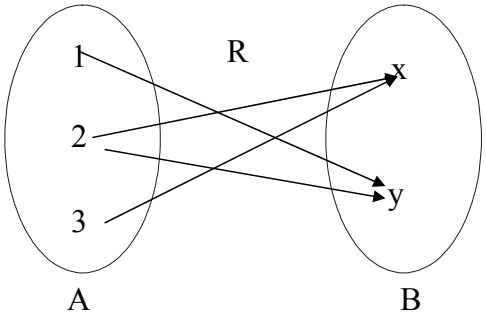
ARROW DIAGRAM OF A RELATION:

Let

A = {1, 2, 3}, B = {x, y} and R = {1,y), (2,x), (2,y), (3,x)}

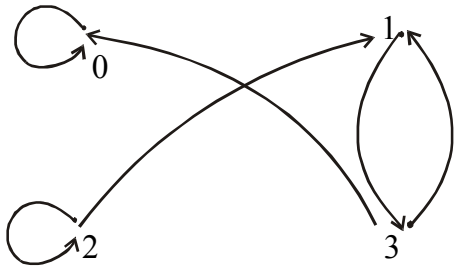
be a relation from A to B.

The arrow diagram of R is:



DIRECTED GRAPH OF A RELATION:

Let A = {0, 1, 2, 3} and R = {(0,0), (1,3), (2,1), (2,2), (3,0), (3,1)}
be a binary relation on A.



DIRECTED GRAPH

MATRIX REPRESENTATION OF A RELATION

Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$. Let R be a relation from A to B. Define the n × m order matrix M by

$$m(i,j) = \begin{cases} 1 \text{ if } (a_i, b_i) \in R \\ 0 \text{ if } (a_i, b_i) \notin R \end{cases}$$

for i=1,2,…,n and j=1,2,…,m

EXAMPLE:

Let A = {1, 2, 3} and B = {x, y}

Let R be a relation from A to B defined as

R ={(1,y), (2,x), (2,y), (3,x)}

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}_{3\times 2} \end{array}$$

EXAMPLE:

For the relation matrix.

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

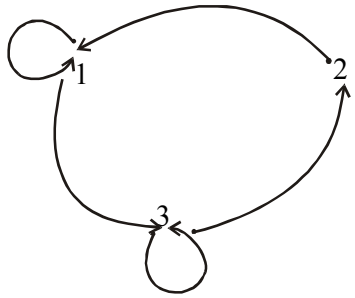
1. List the set of ordered pairs represented by M.
2. Draw the directed graph of the relation.

SOLUTION:

The relation corresponding to the given Matrix is

- R = {(1,1), (1,3), (2,1), (3,2), (3,3)}

And its Directed graph is given below



EXERCISE:

Let A = {2, 4} and B = {6, 8, 10} and define relations R and S from A to B as follows:

for all (x,y) $\in$ A × B, x R y $\Leftrightarrow$ x | y

for all (x,y) $\in$ A × B, x S y $\Leftrightarrow$ y – 4 = x

State explicitly which ordered pairs are in A × B, R, S, R$\cup$S and R$\cap$S.

SOLUTION

A × B = {(2,6), (2,8), (2,10), (4,6), (4,8), (4,10)}

R = {(2,6), (2,8), (2,10), (4,8)}

S = {(2,6), (4,8)}

R $\cup$ S = {(2,6), (2,8), (2,10), (4,8)}= R

R $\cap$ S = {(2,6), (4,8)}= S

LECTURE # 9

Sets Continued

REFLEXIVE RELATION:

Let R be a relation on a set A. R is reflexive if, and only if, for all $a \in A$, $(a, a) \in R$. Or equivalently aRa.

That is, each element of A is related to itself.

REMARK

R is not reflexive iff there is an element "a" in A such that $(a, a) \notin R$. That is, some element "a" of A is not related to itself.

EXAMPLE:

Let A = {1, 2, 3, 4} and define relations R_1,R_2, R_3, R_4 on A as follows:

$R_1 = \{(1, 1), (3, 3), (2, 2), (4, 4)\}$

$R_2 = \{(1, 1), (1, 4), (2, 2), (3, 3), (4, 3)\}$

$R_3 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

$R_4 = \{(1, 3), (2, 2), (2, 4), (3, 1), (4, 4)\}$

Then,

R_1 is reflexive, since $(a, a) \in R_1$ for all $a \in A$.

R_2 is not reflexive, because $(4, 4) \notin R_2$.

R_3 is reflexive, since $(a, a) \in R_3$ for all $a \in A$.

R_4 is not reflexive, because $(1, 1) \notin R_4$, $(3, 3) \notin R_4$

DIRECTED GRAPH OF A REFLEXIVE RELATION:

The directed graph of every reflexive relation includes an arrow from every point to the point itself (i.e., a loop).

EXAMPLE :

Let A = {1, 2, 3, 4} and define relations R_1, R_2, R_3, and R_4 on A by

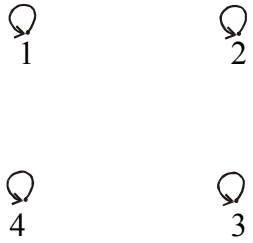
$R_1 = \{(1, 1), (3, 3), (2, 2), (4, 4)\}$

$R_2 = \{(1, 1), (1, 4), (2, 2), (3, 3), (4, 3)\}$

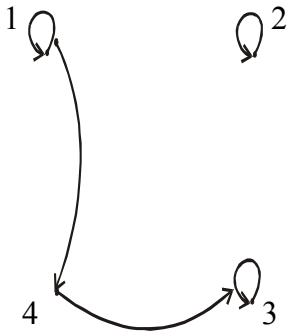
$R_3 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

$R_4 = \{(1, 3), (2, 2), (2, 4), (3, 1), (4, 4)\}$

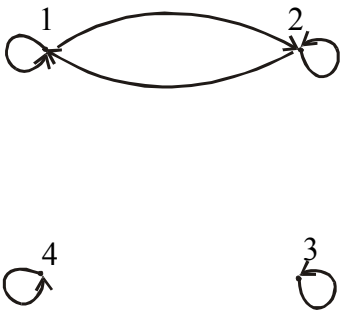
Then their directed graphs are



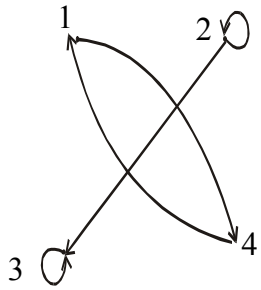
R_1 is reflexive because at every point of the set A we have a loop in the graph.



R_2 is not reflexive, as there is no loop at 4.



R_3 is reflexive



R_4 is not reflexive, as there are no loops at 1and 3.

MATRIX REPRESENTATION OF A REFLEXIVE RELATION:

Let $A = \{a_1, a_2, \ldots, a_n\}$. A Relation R on A is reflexive if and only if $(a_i, a_j) \in R \ \forall \ i=1,2, \ldots,n$.

Accordingly, R is **reflexive** if all the elements on the **main diagonal** of the matrix **M** representing R are equal to 1.

EXAMPLE:

The relation R = {(1,1), (1,3), (2,2), (3,2), (3,3)} on A = {1,2,3} represented by the following matrix M, is reflexive.

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

SYMMETRIC RELATION

Let R be a relation on a set A. R is symmetric if, and only if, for all a, b $\in$ A, if (a, b) $\in$R then (b, a) $\in$R.
That is, if aRb then bRa.

REMARK

R is not symmetric iff there are elements *a* and *b* in A such that (a, b) $\in$R but (b, a) $\notin$R.

EXAMPLE

Let A = {1, 2, 3, 4} and define relations R_1, R_2, R_3, and R_4 on A as follows.

$R_1 = \{(1, 1), (1, 3), (2, 4), (3, 1), (4,2)\}$
$R_2 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$
$R_3 = \{(2, 2), (2, 3), (3, 4)\}$
$R_4 = \{(1, 1), (2, 2), (3, 3), (4, 3), (4, 4)\}$

Then R_1 is symmetric because for every order pair (a,b)in R_1awe have (b,a) in $R_1$for example we have (1,3)in R_1 the we have (3,1) in R_1 similarly all other ordered pairs can be cheacked.

R_2 is also symmetric symmetric we say it is vacuously true.

R_3 is not symmetric, because $(2,3) \in R_3$ but $(3,2) \notin R_3$.

R_4 is not symmetric because $(4,3) \in R_4$ but $(3,4) \notin R_4$.

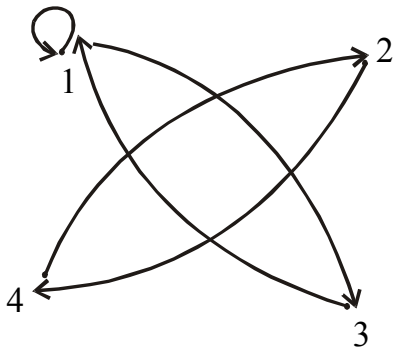
DIRECTED GRAPH OF A SYMMETRIC RELATION

For a symmetric directed graph whenever there is an arrow going from one point of the graph to a second, there is an arrow going from the second point back to the first.

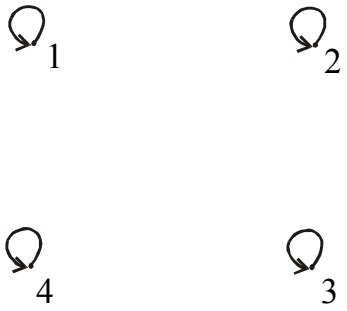
EXAMPLE

Let A = {1, 2, 3, 4} and define relations R_1, R_2, R_3, and R_4 on A by the directed graphs:

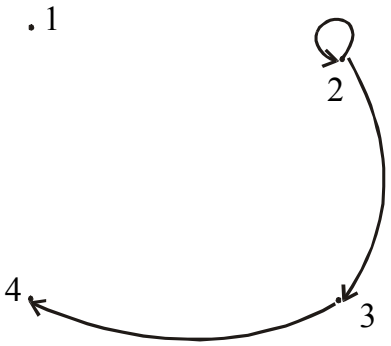
$R_1 = \{(1, 1), (1, 3), (2, 4), (3, 1), (4,2)\}$
$R_2 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$
$R_3 = \{(2, 2), (2, 3), (3, 4)\}$
$R_4 = \{(1, 1), (2, 2), (3, 3), (4, 3), (4, 4)\}$



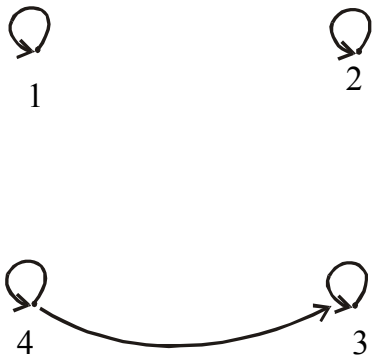
R_1 is symmetric



R_2 is symmetric



R_3 is not symmetric since there are arrows from 2 to 3 and from 3 to 4 but not conversely



R_4 is not symmetric since there is an arrow from 4 to 3 but no arrow from 3 to 4

MATRIX REPRESENTATION OF A SYMMETRIC RELATION

Let

$$A = \{a_1, a_2, \ldots, a_n\}.$$

A relation R on A is symmetric if and only if for all $a_i, a_j \in A$, if $(a_i, a_j) \in R$ then $(a_j, a_i) \in R$.

Accordingly, R is symmetric if the elements in the ith row are the same as the elements in the ith column of the matrix M representing R. More precisely, M is a symmetric matrix.i.e. $M = M^t$

EXAMPLE

The relation R = {(1,3), (2,2), (3,1), (3,3)} on A = {1,2,3} represented by the following matrix M is symmetric.

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

TRANSITIVE RELATION

Let R be a relation on a set A.R is transitive if and only if for all a, b, c ∈A,
if (a, b) ∈R and (b, c) ∈R then (a, c) ∈R.
That is, if aRb and bRc then aRc.

In words, if any one element is related to a second and that second element is related to a third, then the first is related to the third.

Note that the "first", "second" and "third" elements need not to be distinct.

REMARK

R is not transitive iff there are elements a, b, c in A such that
If (a,b) ∈R and (b,c) ∈R but (a,c) ∉R.

(a,B) (b,c)(c,a)

EXAMPLE

Let A = {1, 2, 3, 4} and define relations R_1, R_2 and R_3 on A as follows:

$R_1 = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$
$R_2 = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$
$R_3 = \{(2, 1), (2, 4), (2, 3), (3,4)\}$

Then R_1 is transitive because (1, 1), (1, 2) are in R then to be transitive relation (1,2) must be there and it belongs to R Similarly for other order pairs.

R_2 is not transitive since (1,2) and (2,3) ∈ R_2 but (1,3) ∉ R_2.

R_3 is transitive.

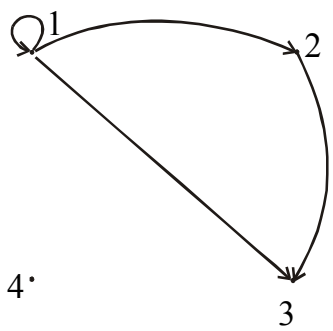
DIRECTED GRAPH OF A TRANSITIVE RELATION

For a transitive directed graph, whenever there is an arrow going from one point to the second, and from the second to the third, there is an arrow going directly from the first to the third.

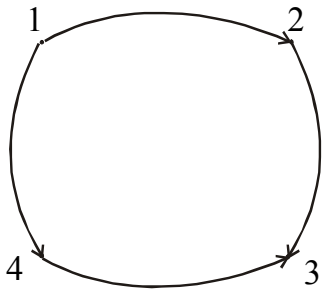
EXAMPLE

Let A = {1, 2, 3, 4} and define relations R_1, R_2 and R_3 on A by the directed graphs:

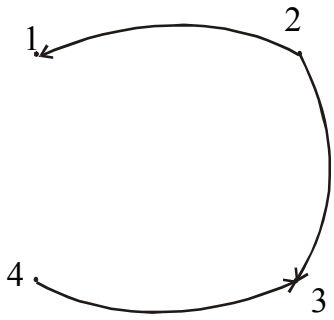
$R_1 = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$
$R_2 = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$
$R_3 = \{(2, 1), (2, 4), (2, 3), (3,4)\}$



R_1 is transitive



R_2 is not transitive since there is an arrow from 1 to 2 and from 2 to 3 but no arrow from 1 to 3 directly



R_3 is transitive

EXERCISE:

Let A = {1, 2, 3, 4} and define the null relation ϕ and universal relation A ×A on A. Test these relations for reflexive, symmetric and transitive properties.

SOLUTION:

Reflexive

(i) $\varnothing$ is not reflexive since (1,1), (2,2), (3,3), (4,4) $\notin \varnothing$.

(ii) $A \times A$ is reflexive since $(a,a) \in A \times A$ for all $a \in A$.

Symmetric

(i) For the null relation $\varnothing$ on A to be symmetric, it must satisfy the implication:

if $(a,b) \in \varnothing$ then $(a, b) \in \varnothing$.

Since $(a, b) \in \varnothing$ is never true, the implication is vacuously true or true by default.

Hence $\varnothing$ is symmetric.

(ii) The universal relation $A \times A$ is symmetric, for it contains all ordered pairs of elements of A. Thus,

if $(a, b) \in A \times A$ then $(b, a) \in A \times A$ for all a, b in A.

Transitive

(i) The null relation $\varnothing$ on A is transitive, because the implication.

if $(a, b) \in \varnothing$ and $(b, c) \in \varnothing$ then $(a, c) \in \varnothing$ is true by default,

since the condition $(a, b) \in \varnothing$ is always false.

(i) The universal relation $A \times A$ is transitive for it contains all ordered pairs of elements of A.

Accordingly, if $(a, b) \in A \times A$ and $(b, c) \in A \times A$ then $(a, c) \in A \times A$ as well.

EXERCISE:

Let $A = \{0, 1, 2\}$ and

$R = \{(0,2), (1,1), (2,0)\}$ be a relation on A.

1. Is R reflexive? Symmetric? Transitive?
2. Which ordered pairs are needed in R to make it a reflexive and transitive relation.

SOLUTION:

1. R is not reflexive, since $0 \in A$ but $(0, 0) \notin R$ and also $2 \in A$ but $(2, 2) \notin R$.
   R is clearly symmetric.

R is not transitive, since $(0, 2)$ & $(2, 0) \in R$ but $(0, 0) \notin R$.

2. For R to be reflexive, it must contain ordered pairs (0,0) and (2,2).

For R to be transitive,

we note (0,2) and (2,0) $\in$ but (0,0) $\notin R$.

Also (2,0) and (0,2) $\in R$ but (2,2) $\notin R$.

Hence (0,0) and (2,2). Are needed in R to make it a transitive relation.

EXERCISE:

Define a relation L on the set of real numbers **R** be defined as follows:

for all $x, y \in R$, $x \, L \, y \Leftrightarrow x < y$.

a. Is L reflexive?
b. Is L symmetric?
c. Is L transitive?

SOLUTION:

a. L is not reflexive, because $x < x$ for any real number x.
   (e.g. $1 \nless 1$)

b. L is not symmetric, because for all $x, y \in R$, if
   $x < y$ then $y \nless x$
   (e.g. $0 < 1$ but $1 \nless 0$)

c. L is transitive, because for all, $x, y, z \in R$, if $x < y$ and $y < z$, then $x < z$.
   (by transitive law of order of real numbers).

EXERCISE:

Define a relation R on the set of positive integers Z^+ as follows:

for all $a, b \in Z^+$, $a \, R \, b$ *iff* $a \times b$ is odd.

Determine whether the relation is

a. reflexive b. symmetric c. transitive

SOLUTION:

Firstly, recall that the product of two positive integers is odd if and only if both of them are odd.

a. reflexive

R is not reflexive, because $2 \in Z^+$ but $2 \, R \, 2$ for $2 \times 2 = 4$ which is not odd.

b. symmetric

R is symmetric, because

if $a \, R \, b$ then $a \times b$ is odd or equivalently $b \times a$ is odd

$(b \times a = a \times b) \Rightarrow b \, R \, a$.

c. transitive

R is transitive, because if $a \, R \, b$ then $a \times b$ is odd

$\Rightarrow$ both "a" and "b" are odd. Also bRc means $b \times c$ is odd

$\Rightarrow$ both "b" and "c" are odd.

Now if aRb and bRc, then all of a, b, c are odd and so $a \times c$ is odd. Consequently aRc.

EXERCISE:

Let "D" be the "divides" relation on Z defined as:

for all m, n $\in$Z, m D n$\Leftrightarrow$ m|n

Determine whether D is reflexive, symmetric or transitive. Justify your answer.

SOLUTION:

Reflexive

Let m $\in$Z, since every integer divides itself so m|m $\forall$ m $\in$Z therefore m D m $\forall$ m $\in$Z

Accordingly D is reflexive

Symmetric

Let m, n $\in$ Z and suppose m D n.

By definition of D, this means m|n (i.e.= an integer)

Clearly, then it is not necessary that = an integer.

Accordingly, if m D n then n D m, $\forall$ m, n $\in$Z

Hence D is not symmetric.

Transitive

Let m, n, p $\in$Z and suppose m D n and n D p.

Now m D n $\Rightarrow$ m|n $\Rightarrow$ = an integer.

Also n D p $\Rightarrow$ n|p $\Rightarrow$ = an integer.

We note = $\frac{p}{m}$ * $\left(\frac{p}{n}\right)$(an int) * $\left(\frac{n}{m}\right)$(an int)

= an int

$\Rightarrow$ m|p and so mDp

Thus if mDn and nDp then mDp $\forall$ m, n, p $\in$Z

Hence D is transitive.

EXERCISE

Let A be the set of people living in the world today. A binary relation R is defined on A as follows:

for all p, q $\in$A, pRq $\Leftrightarrow$ p has the same first name as q.

Determine whether the relation R is reflexive, symmetric and/or transitive.

SOLUTION

a. **Reflexive**

Since every person has the same first name as his/her self. Hence for all p $\in$ A, pRp. Thus, R is reflexive.

b. **Symmetric:**

Let p, q $\in$A and suppose pRq.

$\Leftrightarrow$ p has the same first name as q.

$\Leftrightarrow$ q has the same first name as p.

$\Leftrightarrow$ q R p

Thus if pRq then qRp $\forall$ p,q $\in$A.

$\Rightarrow$ R is symmetric.

c. **Transitive**

Let p, q, s $\in$A and suppose p R q and qRr.

Now pRq $\Leftrightarrow$p has the same first name as q

and qRr $\Leftrightarrow$ q has the same first name as r.

Consequently, p has the same first name as r.

$\Leftrightarrow$ pRr

Thus, if pRq and qRs then pRr, $\forall$ p, q, r $\in$A.

Hence R is transitive.

EQUIVALENCE RELATION

Let A be a non-empty set and R a binary relation on A. R is an equivalence relation if, and only if, R is reflexive, symmetric, and transitive.

EXAMPLE

Let A = {1, 2, 3, 4} and R = {(1,1), (2,2), (2,4), (3,3), (4,2), (4,4)}
be a binary relation on A.
Note that R is reflexive, symmetric and transitive, hence an equivalence relation.

CONGRUENCES

Let m and n be integers and d be a positive integer. The notation

$m \equiv n \pmod d$ means that

d | (m – n) {d divides m minus n}.There exists an integer k such that

$(m - n) = d \cdot k$

EXAMPLE

c. Is $22 \equiv 1 \pmod 3$? b. Is $-5 \equiv +10 \pmod 3$?

d. Is $7 \equiv 7 \pmod 3$? d. Is $14 \equiv 4 \pmod 3$?

SOLUTION

a. Since 22-1 = 21 = 3×7.
Hence 3|(22-1), and so $22 \equiv 1 \pmod 3$

b. Since – 5 – 10 = - 15 = 3 × (-5),
Hence 3|((-5)-10), and so $-5 \equiv 10 \pmod 3$

c. Since 7 – 7 = 0 = 3 × 0
Hence 3|(7-7), and so $7 \equiv 7 \pmod 3$

d. Since 14 – 4 = 10, and 3 / 10 because $10 \neq 3 \cdot k$ for any integer k. Hence $14 \equiv 4 \pmod 3$.

EXERCISE

Define a relation R on the set of all integers Z as follows:

for all integers m and n, m R n $\Leftrightarrow$ $m \equiv n \pmod 3$

Prove that R is an equivalence relation.

SOLUTION

1. **R is reflexive.**

R is reflexive iff for all m $\in$Z, m R m.
By definition of R, this means that

For all m $\in$Z, $m \equiv m \pmod 3$

Since m – m = 0 = 3 ×0.

Hence 3|(m-m), and so $m \equiv m \pmod 3$

$\Leftrightarrow$ mRm

$\Rightarrow$ R is reflexive.

2. **R is symmetric.**

R is symmetric iff for all m, n $\in$Z
if m R n then n R m.

Now mRn $\Rightarrow$ $m \equiv n \pmod 3$

$\Rightarrow$ 3|(m-n)

$\Rightarrow$ m-n = 3k, for some integer k.

$\Rightarrow$ n – m = 3(-k), -k $\in$Z

$\Rightarrow$ 3|(n-m)

$\Rightarrow$ $n \equiv m \pmod 3$

$\Rightarrow$ nRm

Hence R is symmetric.

3. **R is transitive**

R is transitive iff for all m, n, p $\in$Z,
if mRn and nRp then mRp

Now mRn and nRp means $m \equiv n \pmod 3$ and $n \equiv p \pmod 3$

$\Rightarrow$ 3|(m-n) and 3|(n-p)

$\Rightarrow$ (m-n) = 3r and (n-p) = 3s for some r, s $\in$Z

Adding these two equations, we get,

$(m – n) + (n – p) = 3r + 3s$

$\Rightarrow$ $m – p = 3(r + s)$, where $r + s \in Z$

$\Rightarrow$ $3|(m – p)$

$\Rightarrow$ $m \equiv p \pmod 3 \Leftrightarrow m \, R p$

Hence R is transitive. R being reflexive, symmetric and transitive, is an equivalence relation.

LECTURE # 10

Sets Continued

EXERCISE

Suppose R and S are binary relations on a set A.

a. If R and S are reflexive, is R ∩ S reflexive?
b. If R and S are symmetric, is R ∩ S symmetric?
c. If R and S are transitive, is R ∩ S transitive?

SOLUTION

a. R ∩ S is reflexive

Suppose R and S are reflexive. (a,a)(b,b)(c,c)

Then by definition of reflexive relation

∀ a ∈A (a,a) ∈R and (a,a) ∈S

⇒ ∀ a ∈ A (a,a) ∈ R ∩ S

(by definition of intersection)

Accordingly, R ∩ S is reflexive.

b. R ∩ S is symmetric (a,b)(b,a)

Suppose R and S are symmetric.

To prove R ∩ S is symmetric we need to show that

∀ a, b ∈ A, if (a,b) ∈ R ∩ S then (b,a) ∈ R ∩ S.

Suppose (a,b) ∈ R ∩ S.

⇒ (a,b) ∈ R and (a,b) ∈ S

(by the definition of Intersection of two sets)

Since R is symmetric, therefore if (a,b) ∈ R then

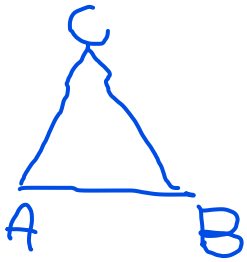
(b,a) ∈ R. Similarly S is symmetric, so if (a,b) ∈ S then (b,a) ∈ S.

Thus (b,a) ∈ R and (b,a) ∈ S

⇒ (b,a) ∈ R ∩ S (by definition of intersection)

Accordingly, R ∩ S is symmetric.

c. R∩S is transitive (a,b)(b,c)(a,c)



Suppose R and S are transitive.

To prove R∩S is transitive we must show that

∀ a,b,c, ∈A, if (a,b) ∈ R∩S and (b,c) ∈ R∩S

then (a,c) ∈R∩S.

Suppose (a,b) ∈R∩S and (b,c) ∈R∩S

⇒ (a,b) ∈R and (a,b) ∈S and (b,c) ∈R and (b,c) ∈S

Since R is transitive, therefore

if (a,b) ∈R and (b,c) ∈R then (a,c) ∈R.

Also S is transitive, so (a,c) ∈S

Hence we conclude that (a,c) ∈R and (a,c) ∈S

and so (a,c) ∈R∩S (by definition of intersection)

Accordingly, R∩S is transitive.

EXAMPLE:

Let A = {1,2,3,4} and let R and S be transitive binary relations on A defined as:

R = {(1,2), (1,3), (2,2), (3,3), (4,2), (4,3)}

and S = {(2,1), (2,4),(3,3)}

Then R ∪ S = {(1,2), (1,3), (2,1), (2,2), (2,4), (3,3), (4,2), (4,3)}

We note (1,2) and (2,1) ∈R∪S, but (1,1) ∉ R∪S

Hence R∪S is not transitive.

IRREFLEXIVE RELATION

Let R be a binary relation on a set A. R is irreflexive iff for all a∈A,(a,a) ∉R. That is, R is irreflexive if no element in A is related to itself by R.

REMARK

R is not irreflexive iff there is an element $a \in A$ such that $(a,a) \in R$.

EXAMPLE

Let A = {1,2,3,4} and define the following relations on A:

$R_1 = \{(1,3), (1,4), (2,3), (2,4), (3,1), (3,4)\}$

$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$

$R_3 = \{(1,2), (2,3), (3,3), (3,4)\}$

Then R_1 is irreflexive since no element of A is related to itself in R_1. i.e.

$(1,1) \notin R_1$, $(2,2) \notin R_1$, $(3,3) \notin R_1$, $(4,4) \notin R_1$

R_2 is not irreflexive, since all elements of A are related to themselves in R_2

R_3 is not irreflexive since $(3,3) \in R_3$. Note that R_3 is not reflexive.

NOTE

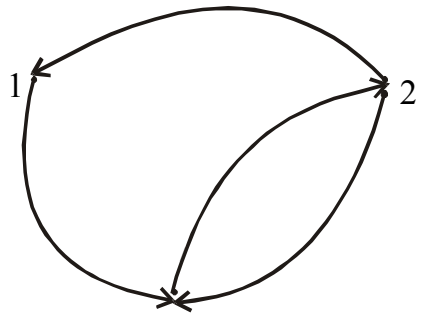
A relation may be neither reflexive nor irreflexive.

DIRECTED GRAPH OF AN IRREFLEXIVE RELATION

Let R be an irreflexive relation on a set A. Then by definition, no element of A is related to itself by R. Accordingly, there is no loop at each point of A in the directed graph of R.

EXAMPLE

Let A = {1,2,3} and R = {(1,3), (2,1), (2,3), (3,2)} be represented by the directed graph.



MATRIX REPRESENTATION OF AN IRREFLEXIVE RELATION

Let R be an irreflexive relation on a set A. Then by definition, no element of A is related to itself by R.

Since the self related elements are represented by 1's on the main diagonal of the matrix representation of the relation, so for irreflexive relation R, the matrix will contain all 0's in its main diagonal.

It means that a relation is irreflexive if in its matrix representation the diagonal elements are all zero, if one of them is not zero the we will say that the relation is not irreflexive.

EXAMPLE:

Let A = {1,2,3} and R = {(1,3), (2,1), (2,3), (3,2)} be represented by the matrix

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Then R is irreflexive, since all elements in the main diagonal are 0's.

EXERCISE

Let R be the relation on the set of integers Z defined as:

for all $a,b \in Z$, $(a,b) \in R \Leftrightarrow a > b$.

Is R irreflexive?

SOLUTION

R is irreflexive if for all $a \in Z$, $(a,a) \notin R$.

Now by the definition of given relation R,for all $a \in Z$, $(a,a) \notin R$ since $a > a$.

Hence R is irreflexive.

ANTISYMMETRIC RELATION

Let R be a binary relation on a set A.R is anti-symmetric iff

$\forall a, b \in A$ if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$.

REMARK

1) R is not **anti-symmetric** iff there are elements a and b in A such that $(a,b) \in R$ and $(b,a) \in R$ but $a \neq b$.
2) The properties of being symmetric and being anti-symmetric are not negative of each other.

EXAMPLE

Let $A = \{1,2,3,4\}$ and define the following relations on A.

$R_1 = \{(1,1),(2,2),(3,3)\}$ $R_2 = \{(1,2),(2,2), (2,3), (3,4), (4,1)\}$

$R_3=\{(1,3),(2,2), (2,4), (3,1), (4,2)\}$ $R_4=\{(1,3),(2,4), (3,1), (4,3)\}$

R_1 is anti-symmetric and symmetric .

R_2 is anti-symmetric but not symmetric because $(1,2) \in R_2$but $(2,1) \notin R_2$.

R_3 is not anti-symmetric since $(1,3)$ & $(3,1) \in R_3$ but $1 \neq 3$.

Note that R_3 is symmetric.

$R_4$is neither anti-symmetric because $(1,3)$ & $(3,1) \in R_4$ but $1 \neq 3$ nor symmetric because $(2,4) \in R_4$ but $(4,2) \notin R_4$

DIRECTED GRAPH OF AN ANTISYMMETRIC RELATION

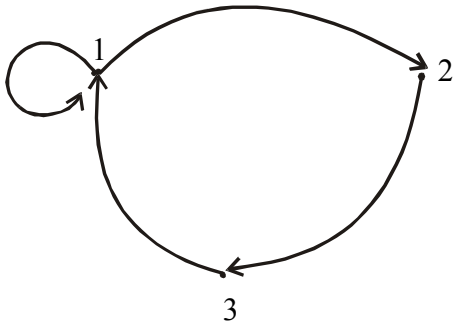
Let R be an anti-symmetric relation on a set A. Then by definition, no two distinct elements of A are related to each other.

Accordingly, there is no pair of arrows between two distinct elements of A in the directed graph of R.

EXAMPLE

Let A = {1,2,3} And R be the relation defined on A is

R ={(1,1), (1,2), (2,3), (3,1)}.Thus R is represented by the directed graph as



R is anti-symmetric, since there is no pair of arrows between two distinct points in A.

MATRIX REPRESENTATION OF AN ANTISYMMETRIC RELATION

Let R be an anti-symmetric relation on a set

$A = \{a_1, a_2, \ldots, a_n\}$. Then if $(a_i, a_j) \in R$ for $i \neq j$ then $(a_i, a_j) \notin R$.

Thus in the matrix representation of R there is a 1 in the ith row and jth column iff the jth row and ith column contains 0 vice versa.

EXAMPLE

Let A = {1,2,3} and a relation

R = {(1,1), (1,2), (2,3), (3,1)}on A be represented by the matrix.

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Then R is anti-symmetric as clear by the form of matrix M.

PARTIAL ORDER RELATION

Let R be a binary relation defined on a set A. R is a partial order relation,if and only if, R is **reflexive, antisymmetric**, and **transitive**. The set A together with a partial ordering R is called a **partially ordered** set or **poset.**

EXAMPLE

Let R be the set of real numbers and define the"less than or equal to" on R as follows:

for all real numbers x and y in R.$x \leq y \Leftrightarrow x < y$ or $x = y$

Show that ≤ is a partial order relation.

SOLUTION

≤ is reflexive

For ≤ to be reflexive means that $x \leq x$ for all $x \in R$

But $x \leq x$ means that $x < x$ or $x = x$ and $x = x$ is always true.

Hence under this relation every element is related to itself.

≤ is anti-symmetric

For ≤ to be anti-symmetric means that

$\forall$ x,y ∈R, if $x \leq y$ and $y \leq x$, then $x = y$.

This follows from the definition of ≤ and the trichotomy property, which says that "given any real numbers x and y, exactly one of thefollowing holds:

$x < y$ or $x = y$ or $x > y$"

≤ is transitive

For ≤ to be transitive means that

$\forall$ x,y,z ∈R, if x≤ y and $y \leq z$ then $x \leq z$.

This follows from the definition of ≤ and the transitive property of order of real numbers, which says that "given any real numbers x, y and z,

if $x < y$ and $y < z$ then $x < z$"

Thus ≤ being reflexive, anti-symmetric and transitive is a partial order relation on R.

EXERCISE

Let A be a non-empty set and P(A) the power set of A. Define the "subset" relation, ⊆, as follows:

for all X,Y ∈ P(A), $X \subseteq Y \Leftrightarrow \forall$ x, iff x ∈X then x ∈Y.

Show that ⊆ is a partial order relation.

SOLUTION

1. **⊆ is reflexive**

Let X ∈ P(A). Since every set is a subset of itself, therefore

$X \subseteq X$, $\forall$ X ∈P(A).

Accordingly ⊆ is reflexive. (a,b)(b,c)

2. **⊆ is anti-symmetric**

Let X, Y ∈P(A) and suppose $X \subseteq Y$ and $Y \subseteq X$.Then by definition of equality of two sets it follows that X = Y.

Accordingly, ⊆ is anti-symmetric.

3. **⊆ is transitive**

Let X, Y, Z ∈P(A) and suppose $X \subseteq Y$ and $Y \subseteq Z$.

Then by the transitive property of subsets "if $U \subseteq V$ and $V \subseteq W$ then $U \subseteq W$"it follows $X \subseteq Z$.

Accordingly ⊆ is transitive.

EXERCISE

Let "|" be the "divides" relation on a set A of positive integers. That is, for all a, b ∈A, a|b ⇔ b = k ·a for some integer k.
Prove that | is a partial order relation on A.

SOLUTION

1) "|" is reflexive. [We must show that, ∀ a ∈A, a|a] 2/2=1 3/3=1
Suppose a ∈A. Then a = 1·a and so a|a by definition of divisibility.

2) "|" is anti-symmetric

[We must show that for all a, b ∈A, if a|b and b|a then a=b]
Suppose a|b and b|a
By definition of divides there are integers k_1, and k_2 such that

$b = k_1 \cdot a$ and $a = k_2 \cdot b$ 1.2 2.1

Now $b = k_1 \cdot a$
$= k_1 \cdot (k_2 \cdot b)$ (by substitution)
$= (k_1 \cdot k_2) \cdot b$ b/b

Dividing both sides by b gives

$1 = k_1 \cdot k_2$

Since a, b ∈A, where A is the set of positive integers, so the equations

$b = k_1 \cdot a$ and $a = k_2 \cdot b$

implies that k_1 and k_2 are both positive integers. Now the equation

$k_1 \cdot k_2 = 1$

can hold only when $k_1 = k_2 = 1$

Thus $a = k_2 \cdot b = 1 \cdot b = b$ i.e., a = b

3)"|" is transitive

[We must show that ∀a,b,c∈A if a|b and b|c than a|c]
Suppose a|b and b|c
By definition of divides, there are integers k_1 and k_2 such that

$b = k_1 \cdot a$ and $c = k_2 \cdot b$

Now $c = k_2 \cdot b$
$= k_2 \cdot (k_1 \cdot a)$ (by substitution)
k base 1, 2 power $= (k_2 \cdot k_1) \cdot a$ (by associative law under multiplication)
$= k_3 \cdot a$ where $k_3 = k_2 \cdot k_1$ is an integer
$\Rightarrow$ a|c by definition of divides

Thus "|" is a partial order relation on A.

EXERCISE

Let "R" be the relation defined on the set of integers Z as follows:
for all a, b ∈Z, aRb iff $b = a^r$ for some positive integer r.Show that R is a partial order on Z.

SOLUTION

Let a, b ∈Z and suppose aRb and bRa. Then there are positive integers r and s such that

$b = a^r$ and $a = b^s$

Now, $a = b^s$
$= (a^r)^s$ by substitution
$= a^{rs}$

$\Rightarrow$ rs =1

Since r and s are positive integers, so this equation can hold if, and only if, r =1 and s = 1

and then $a = b^s = b^1 = b$ i.e., $a = b$

Thus R is anti-symmetric.

3. Let a, b, $c \in Z$ and suppose aRb and bRc**.**

Then there are positive integers r and s such that

$b = a^r$ and $c = b^s$

Now $c = b^r$

$= (a^r)^s$ (by substitution)

$= a^{rs} = a^t$ (where t = rs is also a positive integer)

Hence by definition of R, aRc. Therefore, R is transitive.

Accordingly, R is a **partial order** relation on Z.

LECTURE # 11

Relations and Functions

RELATIONS AND FUNCTIONS:

A function **F** from a set X to a set Y is a relation from X to Y that satisfies the following two properties

1.For every element x in X, there is an element y in Y such that $(x,y) \in F$.

In other words every element of X is the first element of some ordered pair of F.

2.For all elements x in X and y and z in Y, if $(x,y) \in F$ and $(x,z) \in F$, then $y = z$

In other words no two distinct ordered pairs in F have the same first element.

EXERCISE

Which of the relations define functions from X = {2,4,5} to Y={1,2,4,6}.

a) $R_1 = \{(2,4), (4,1)\}$

b) $R_2 = \{(2,4), (4,1), (4,2), (5,6)\}$

c) $R_3 = \{(2,4), (4,1), (5,6)\}$

SOLUTION:-

a) R_1 is not a function, because $5 \in X$ does not appear as the first element in any ordered pair in R_1.

b) R_2 is not a function, because the ordered pairs (4,1) and (4,2) have the same first element but different second elements.

c) R_3 defines a function because it satisfy both the conditions of the function that is every element of X is the first element of some order pair and there is no pair which has the same first order pair but different second order pair.

EXERCISE:-

Let A = {4,5,6} and B = {5,6} and define binary relations R and S from A to B as follows:

for all $(x,y) \in A \times B$, $(x,y) \in R \Leftrightarrow x \geq y$

for all $(x,y) \in A \times B$, $xSy \Leftrightarrow 2|(x-y)$

a. Represent R and S as a set of ordered pairs.
b. Indicate whether R or S is a function

SOLUTION:-

Since we are given the relation R contains those order pairs of $A \times B$ which has their first element greater or equal to the second Hence R contains the order pairs.

$R = \{(5,5), (6,5), (6,6)\}$

Similarly S is such a relation which consists of those order pairs for which the difference of first and second elements difference divisible by 2.

Hence $S = \{(4,6), (5,5), (6,6)\}$

b. R is not a function because $4 \in A$ is not related to any element of B.

S clearly defines a function since each element of A is related to a unique element of B.

FUNCTION

A function ***f*** from a set X to a set Y is a ***relationship*** between elements of X and elements of Y such that ***each*** element of X is related to a ***unique*** element of Y, and is denoted $f : X \rightarrow Y$. The set X is called the domain of *f* and Y is called the co-domain of *f*.

NOTE: The unique element y of Y that is related to x by *f* is denoted *f*(x) and is called *f* of x, or the value of *f* at x, or the image of x under *f*

ARROW DIAGRAM OF A FUNCTION

The definition of a function implies that the arrow diagram for a function f has the following two properties:

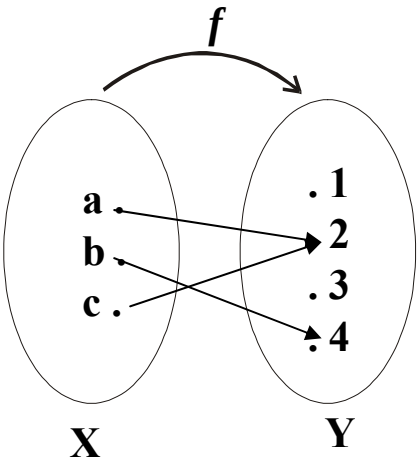
1. Every element of X has an arrow coming out of it

2. No two elements of X has two arrows coming out of it that point to two different elements of Y.

EXAMPLE:-

Let X = {a,b,c} and Y={1,2,3,4}.

Define a function *f* from X to Y by the arrow diagram.

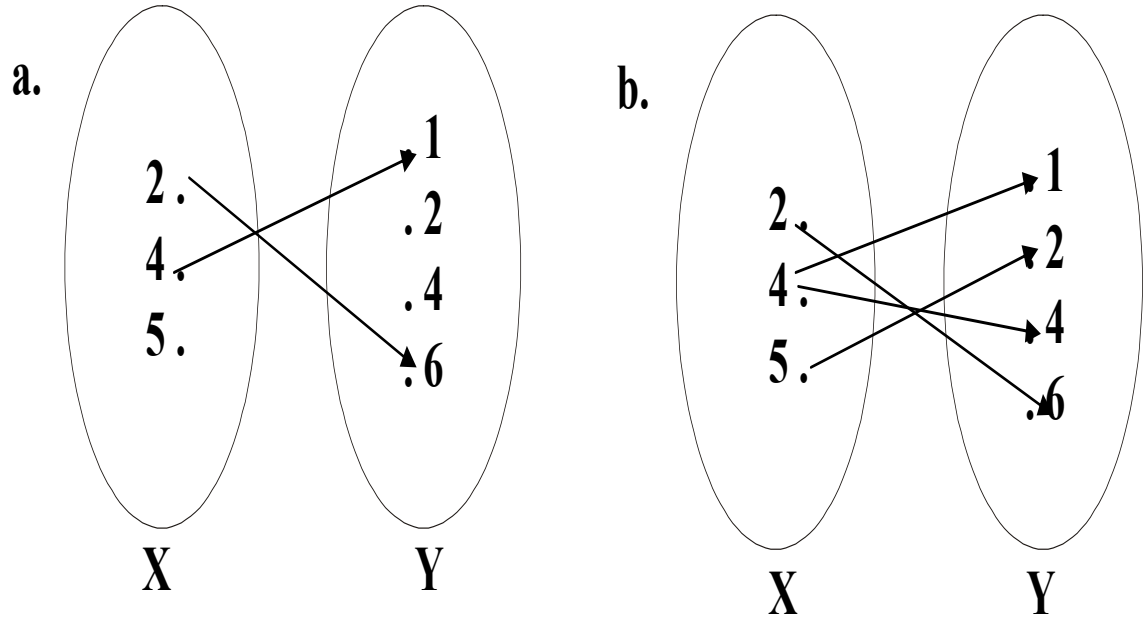


You can easily note that the above diagram satisfy the two conditions of a function hence a graph of the function.

Note that f(a) = 2, f(b) = 4,and f(c) = 2

FUNCTIONS AND NONFUNCTIONS

Which of the arrow diagrams define functions from X = {2,4,5}to Y = {1,2,4,6}.



The relation given in the diagram (a) is **Not a function** because there is no arrow coming out of of $5 \in X$ to any element of Y.

The relation in the diagram (b) is **Not a function**, because there are two arrows coming out of $4 \in X$.

i.e.,$4 \in X$ is not related to a unique element of Y.

RANGE OF A FUNCTION

Let f: $X \rightarrow Y$. The range of f consists of those elements of Y that are image of elements of X.

Symbolically: **Range** of f = $\{y \in Y | y = f(x),$ for some $x \in X\}$

NOTE:

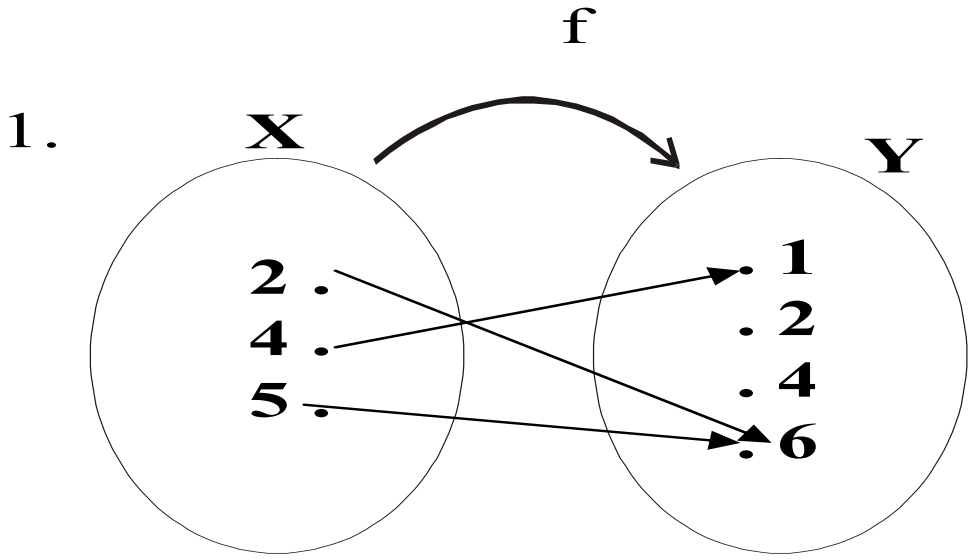
1. The range of a function f is always a subset of the co-domain of f.

2. The range of f: X $\rightarrow$Y is also called the image of X under f.
3. When y = f(x), then x is called the pre-image of y.
4. The set of all elements of X, that are related to some y $\in$Y is called the inverse image of y.

EXERCISE:-

Determine the range of the functions f, g, h from X = {2,4,5} to Y = {1,2,4,6} defined as:

1.



2. g = {(2,6), (4,2), (5,1)}
3. h(2) = 4, h (4) = 4, h(5) = 1

SOLUTION:-

1. Range of f = {1, 6}
2. Range of g = {1, 2, 6}
3. Range of h = {1, 4}

GRAPH OF A FUNCTION

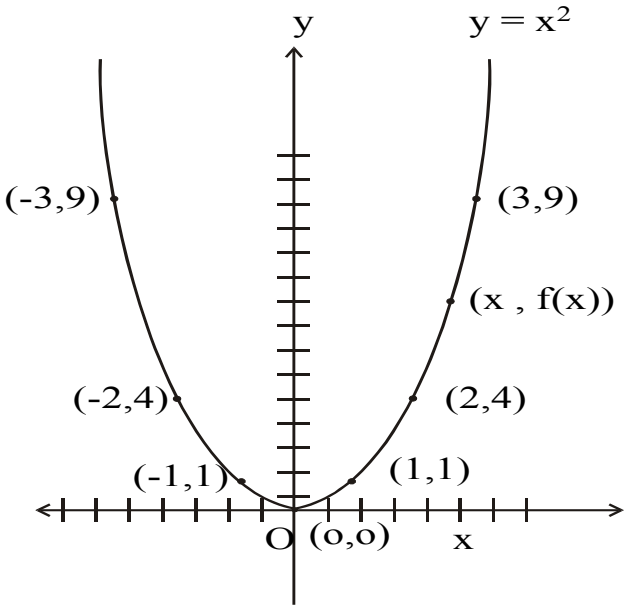
Let f be a real-valued function of a real variable. i.e. f:R $\rightarrow$R. The graph of f is the set of all points (x,y) in the Cartesian coordinate plane with the property that x is in the domain of f and y = f(x).

EXAMPLE:-

We have to draw the graph of the function f given by the relation $y=x^2$ in order to draw the graph of the function we will first take some elements from the domain will see the image of them and then plot then on the graph as follows

Graph of $y = x^2$

x	y=f(x)
-3	9
-2	4
-1	1
0	0
+1	1
+2	4
+3	9

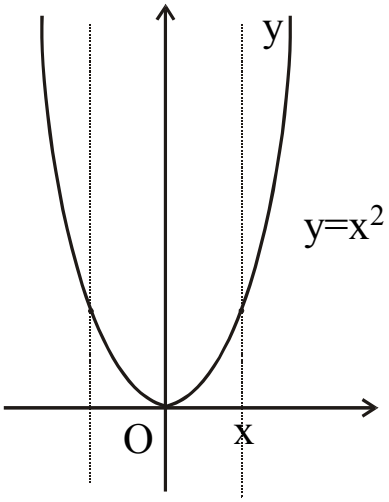


VERTICAL LINE TEST FOR THE GRAPH OF A FUNCTION

For a graph to be the graph of a function, any given vertical line in its domain intersects the graph in at most one point.

EXAMPLE:-

The graph of the relation $y = x^2$ on R defines a function by vertical line test.



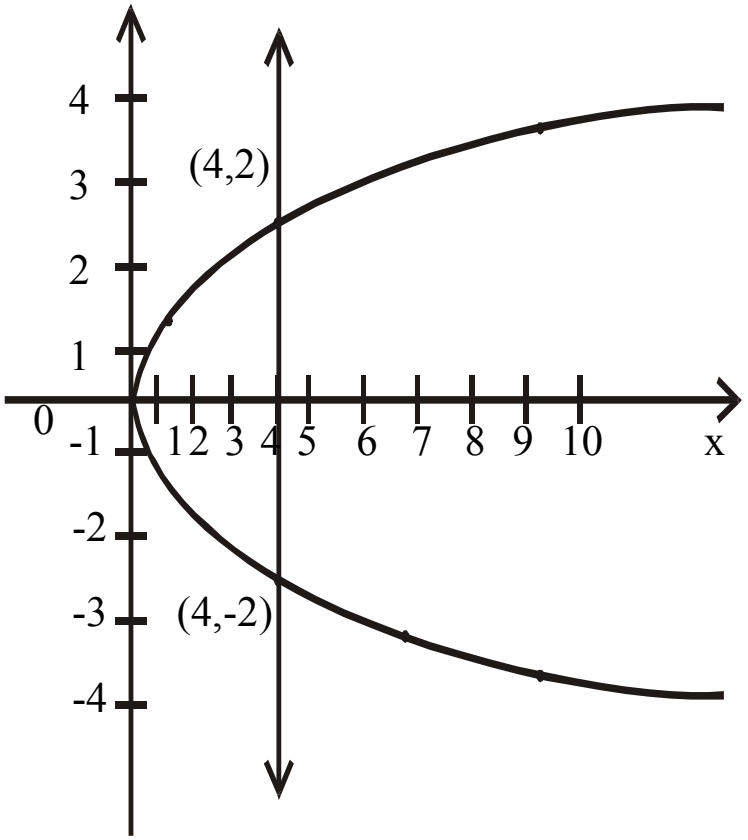
EXERCISE:-

Define a binary relation P from R to R as follows:
for all real numbers x and y $(x, y) \in P \Leftrightarrow x = y^2$
Is P a function? Explain.

SOLUTION:-

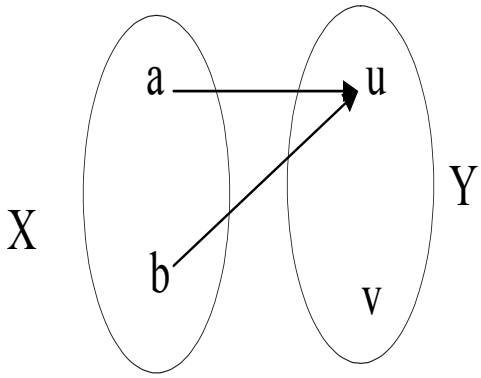
The graph of the relation $x = y^2$ is shown below. Since a vertical line intersects the graph at two points; the graph does not define a function.

x	Y
9	-3
4	-2
1	-1
0	0
1	1
4	2
9	3

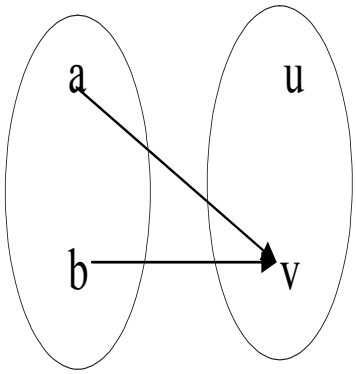


EXERCISE:-
Find all functions from X = {a,b} to Y = {u,v}
SOLUTION:-

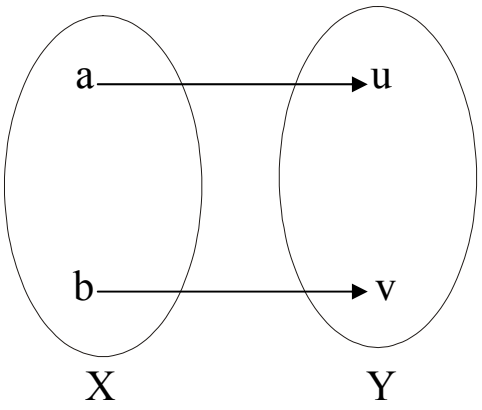
1.



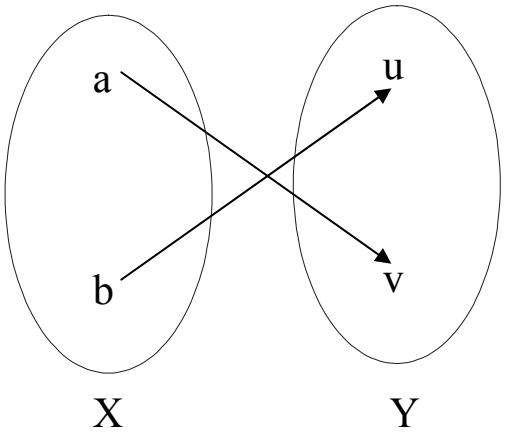
2.



3.

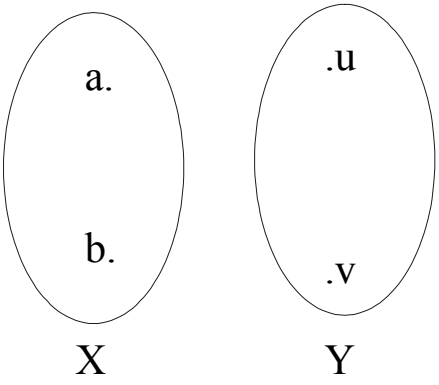


4.

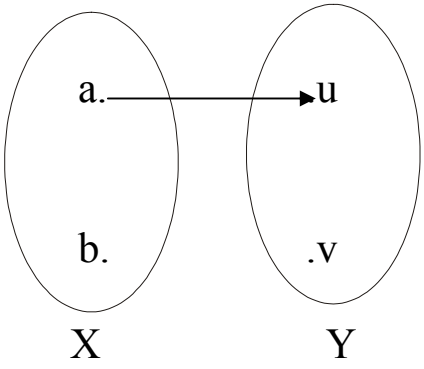


EXERCISE:-
Find four binary relations from X = {a,b}to Y = {u,v}that are not functions.
SOLUTION:-
The four relations are

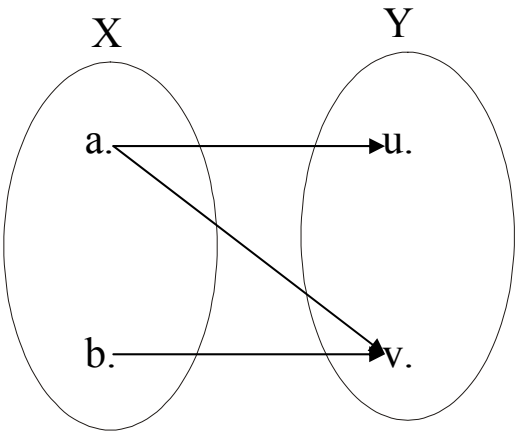
1.



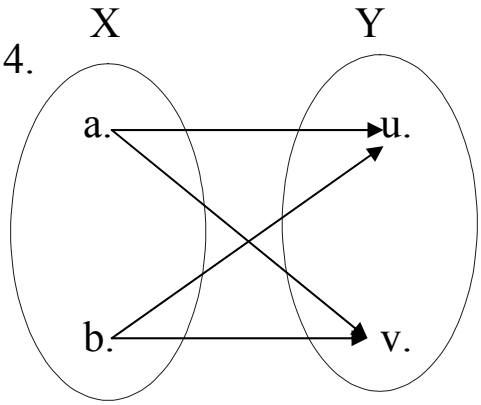
2.



3.



4.



EXERCISE:-
How many functions are there from a set with three elements to a set with four elements.
SOLUTION:-
Let X = $\{x_1, x_2, x_3\}$ and Y= $\{y_1, y_2, y_3, y_4\}$

Then x_1 may be related to any of the four elements y_1, y_2, y_3, y_4 of Y. Hence there are 4 ways to relate x_1 in Y. Similarly x_2 may also be related to any one of the 4 elements in Y. Thus the total number of different ways to relate x_1 and x_2 to elements of Y are $4 \times 4 = 16$. Finally x_3 must also has its image in Y and again any one of the 4 elements y_1, or y_2 or y_3 or y_4 could be its image.
Therefore the total number of functions from X to Y are

$$4 \times 4 \times 4 = 4^3 = 64.$$

EXERCISE:-
Suppose A is a set with m elements and B is a set with n elements.
1.How many binary relations are there from A to B?
2.How many functions are there from A to B?
3.What fraction of the binary relations from A to B are functions?
SOLUTION:-
1.Number of elements in $A \times B = m.n$
Therefore, number of binary relations from A to B =
Number of all subsets of $A \times B = 2^{mn}$
2.Number of functions from A to B = n.n.n. … .n (m times)
$= n^m$
3.Fraction of binary relations that are functions = $n^m / 2^{mn}$
FUNCTIONS NOT WELL DEFINED:-
Determine whether f is a function from Z to R if

a. $f(n) = \pm n$ *b.* $f(n) = \dfrac{1}{n^2 - 4}$

c. $f(n) = \sqrt{n}$ *d.* $f(n) = \sqrt{n^2 + 1}$

SOLUTION:-
a. f is not well defined since each integer n has two images +n and -n
b. f is not well defined since f(2) and f(-2) are not defined.
c. f is not defined for $n < 0$ since f then results in imaginary values (not real)
d. f is well defined because each integer has unique (one and only one) image in R under f.
EXERCISE:-
Student C tries to define a function $h : Q \to Q$ by the rule. $h\left(\dfrac{m}{n}\right) = \dfrac{m^2}{n}$
for all integers m and n with $n \neq 0$
Students D claims that h is not well defined. Justify students D's claim.
SOLUTION:-
The function h is well defined if each rational number has a unique (one and only one) image.

Consider $\dfrac{1}{2} \in Q$

$$h\left(\frac{1}{2}\right) = \frac{1^2}{2} = \frac{1}{2}$$

Now $\dfrac{1}{2} = \dfrac{2}{4}$ and

$$h\left(\frac{2}{4}\right) = \frac{2^2}{4} = \frac{4}{4} = 1$$

Hence an element of Q has more than one images under h. Accordingly h is not well defined.

REMARK:-
A function $f: X \to Y$ is well defined *iff* $\forall\ x_1, x_2 \in X$, if $x_1 = x_2$ then $f(x_1) = f(x_2)$
EXERCISE:-
Let $g: R \to R+$ be defined by $g(x) = x^2 + 1$
1. Show that g is well defined.

2. Determine the domain, co-domain and range of g.

SOLUTION:-

1. g is well defined:

Let $x_1, x_2 \in R$ and suppose $x_1 = x_2$

$\Rightarrow$ $x_1^2 = x_2^2$ (squaring both sides)

$\Rightarrow$ $x_1^2 + 1 = x_2^2 + 1$ (adding 1 on both sides)

$\Rightarrow$ $g(x_1) = g(x_2)$ (by definition of g)

Thus if $x_1 = x_2$ then $g(x_1) = g(x_2)$. According $g: R \rightarrow R+$ is well defined.

2. $g: R \rightarrow R^+$ defined by $g(x) = x^2 + 1$.

Domain of g = R (set of real numbers)

Co-domain of $g = R^+$ (set of positive real numbers)

The range of g consists of those elements of R^+ that appear as image points.

Since $x^2 \geq 0$ $\forall\ x \in R$

$x^2 + 1 \geq 1$ $\forall\ x \in R$

i.e. $g(x) = x^2 + 1 \geq 1\ \forall\ x \in R$

Hence the range of g is all real number greater than or equal to 1, i.e., the internal $[1, \infty)$

IMAGE OF A SET

Let $f : X \rightarrow Y$ is function and $A \subseteq X$.

The image of A under f is denoted and defined as:

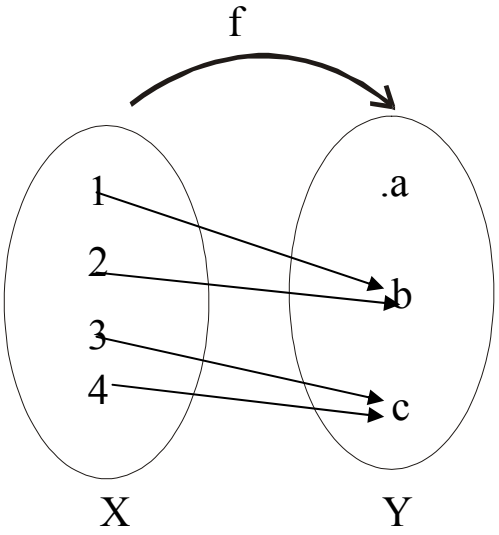
$f(A) = \{y \in Y \mid y = f(x),$ for some x in A$\}$

EXAMPLE:-

Let $f: X \rightarrow Y$ be defined by the arrow diagram

Let A = {1,2}and B = {2,3} then

f(A)={b} and f(B) = {b,c} under the function defined in the Diagram then we say that image set of A is {b} and I mage set of B is {b,c}.



INVERSE IMAGE OF A SET

Let $f: X \rightarrow Y$ is a function and $C \subseteq Y$. The inverse image of C under f is denoted and defined as:

$f^{-1}(C) = \{x \in X \mid f(x) \in C\}$

EXAMPLE:-

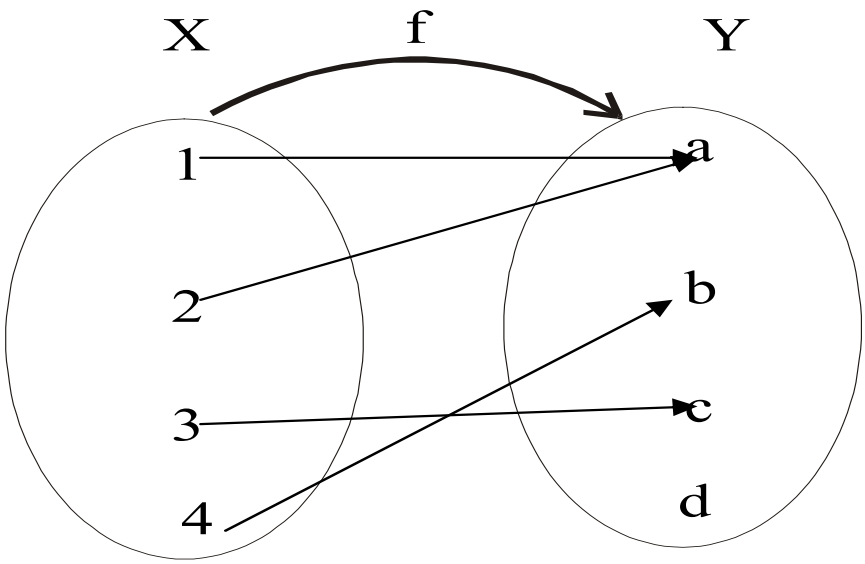
Let $f: X \rightarrow Y$ be defined by the arrow diagram.

Let C = {a},D = {b,c},E = {d} then

$f^{-1}(C) = \{1,2\}$,

$f^{-1}(D) = \{3,4\}$, and

$f^{-1}(E) = \varnothing$



SOME RESULTS:

Let f: X $\rightarrow$Y is a function. Let A and B be subsets of X and C and D be subsets of Y.

1. if A$\subseteq$ B then f(A) $\subseteq$ f(B)
2. f(A$\cup$B) = f(A) $\cup$f(B)
3. f(A$\cap$B) $\subseteq$ f(A) $\cap$ f(B)
4. f(A-B) $\supset$ f(A) - f(B)
5. if C $\subseteq$ D, then $f^{-1}(C) \subseteq f^{-1}(D)$
6. $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
7. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$
8. $f^{-1}(C-D) = f^{-1}(C) - f^{-1}(D)$

BINARY OPERATIONS

A binary operation "*" defined on a set A assigns to each ordered pair (a,b) of elements of A, a uniquely determined element a*b of A.

That is, a binary operation takes two elements of A and maps them to a third element of A.

EXAMPLE:-

1. "+" and "." are binary operations on the set of natural numbers N.
2. "-" is not a binary operation on N.
3. "-" is a binary operation on Z, the set of integers.
4. "÷" is a binary operation on the set of non-zero rational numbers Q-{0}, but not a binary operation on Z.

BINARY OPERATION AS FUNCTION

A binary operation "*" on a set A is a function from A * A to A.

i.e. *: A $\times$A $\rightarrow$A.

Hence *(a,b) = c, where a, b, c $\in$A.

NOTE

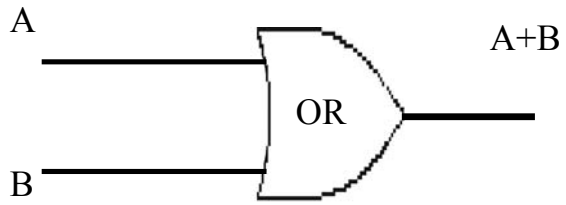
*(a,b) is more commonly written as a*b.

EXAMPLES:-

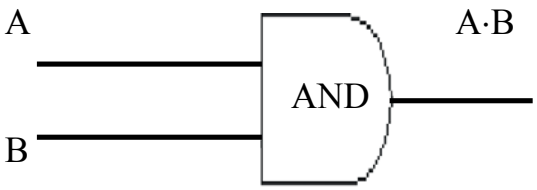
1.The set operations union $\cup$, intersection $\cap$ and set difference -, are binary operators on the power set P(A) of any set A.

2.The logical connectives $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$ are binary operations on the set {T, F}

3**.** The logic gates OR and AND are binary operations on {0,1}

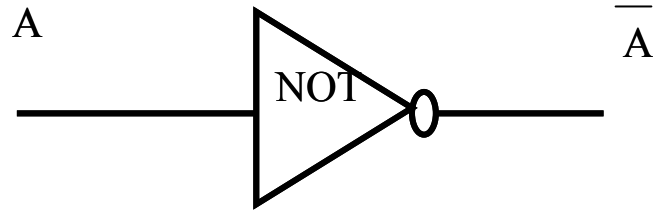


A	B	A+B
1	1	1
1	0	1
0	1	1
0	0	0



A	B	A·B
1	1	1
1	0	0
0	1	0
0	0	0

4. The logic gate NOT is a uniary operation on {0,1}



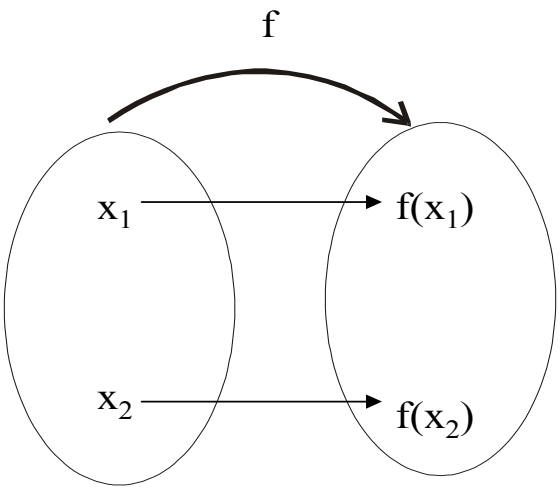
A	$\overline{A}$
1	0
0	1

LECTURE # 12

Functions

INJECTIVE FUNCTION or ONE-TO-ONE FUNCTION

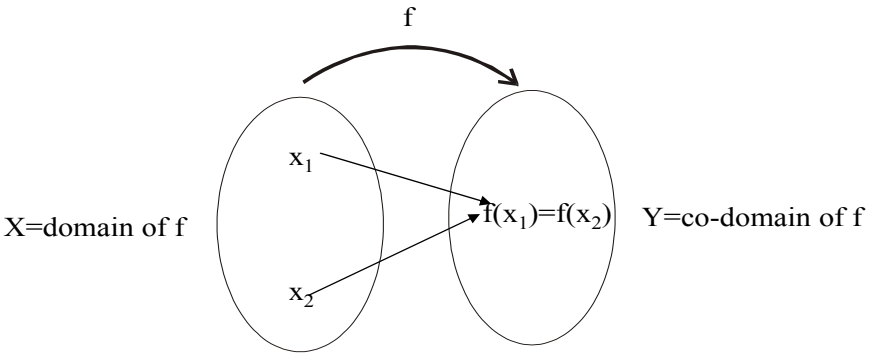
Let f: X $\rightarrow$Y be a function. f is injective or one-to-one if, and only if, $\forall$ x1, x2 $\in$X, if x1 $\neq$ x2 then f(x1) $\neq$ f(x2)That is, f is one-to-one if it maps distinct points of the domain into the distinct points of the co-domain.



A one-to-one function separates points.

FUNCTION NOT ONE-TO-ONE

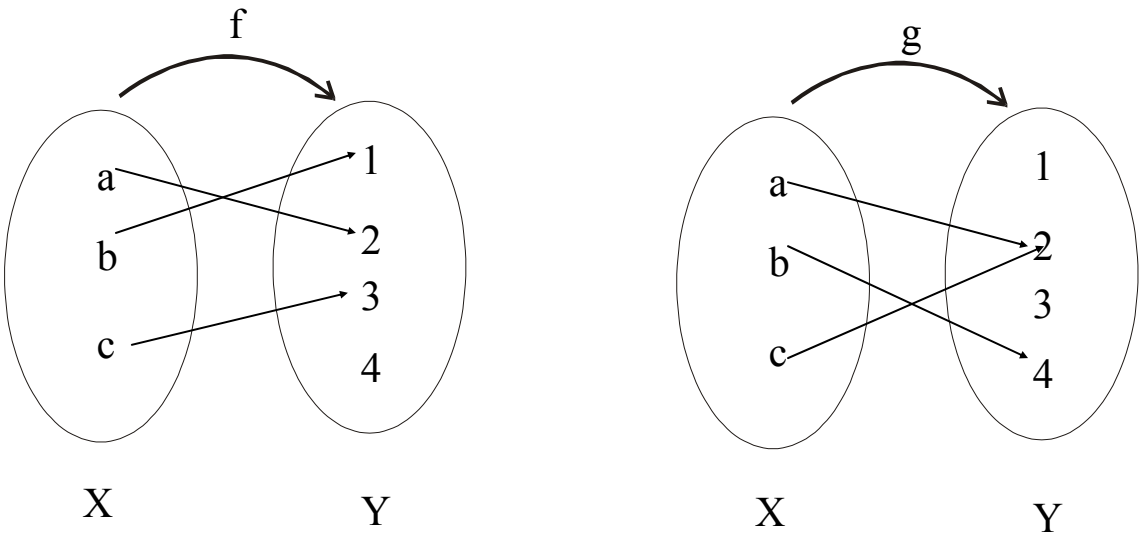
A function f: X $\rightarrow$Y is not one-to-one iff there exist elements x_1 and x_2 in such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$.That is, if distinct elements x_1 and x_2 can found in domain of f that have the same function value.



A function that is not one-to-one collapses points together.

EXAMPLE:-

Which of the arrow diagrams define one-to-one functions?



SOLUTION:-

f is clearly one-to-one function, because no two different elements of X are mapped onto the same element of Y.

g is not one-to-one because the elements a and c are mapped onto the same element 2 of Y.

ALTERNATIVE DEFINITION FOR ONE-TO-ONE FUNCTION

A function f: $X \to Y$ is one-to-one (1-1) iff $\forall\ x_1, x_2 \in X$, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$
(i.e distinct elements of 1st set have their distinct images in 2nd set)

The equivalent contra-positive statement for this implication is $\forall\ x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$

REMARK:-

f: $X \to Y$ is not one-to-one iff $\exists\ x_1, x_2 \in X$ with $f(x_1) = f(x_2)$ but $x_1 \neq x_2$

EXAMPLE:-

Define f: $R \to R$ by the rule $f(x) = 4x - 1$ for all $x \in R$. Is f one-to-one? Prove or give a counter example.

SOLUTION:-

Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$\Rightarrow \quad 4x_1 - 1 = 4x_2 - 1$ (by definition of f)

$\Rightarrow \quad 4x_1 = 4x_2$ (adding 1 to both sides)

$\Rightarrow \quad x_1 = x_2$ (dividing both sides by 4)

Thus we have shown that if $f(x_1) = f(x_2)$ then $x_1 = x_2$

Therefore, f is one-to-one

EXAMPLE:-

Define g : $Z \to Z$ by the rule $g(n) = n^2$ for all $n \in Z$. Is g one-to-one? Prove or give a counter example.

SOLUTION:-

Let $n_1, n_2 \in Z$ and suppose $g(n_1) = g(n_2)$

$\Rightarrow \quad n_1^2 = n_2^2$ (by definition of g)

$\Rightarrow$ either $n_1 = +n_2$ or $n_1 = -n_2$

Thus $g(n_1) = g(n_2)$ does not imply $n_1 = n_2$ always.

As a counter example, let $n_1 = 2$ and $n_2 = -2$.

Then

$g(n_1) = g(2) = 2^2 = 4$ and also $g(n_2) = g(-2) = (-2)^2 = 4$

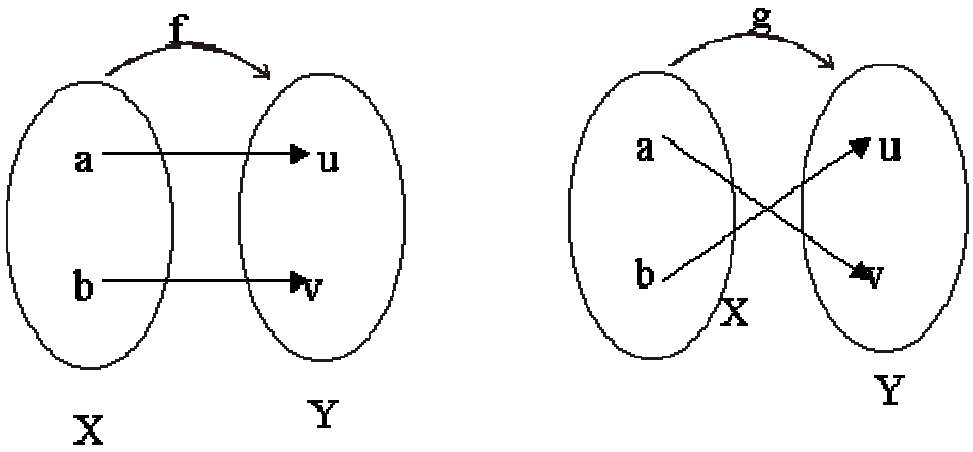
Hence $g(2) = g(-2)$ where as $2 \neq -2$ and so g is not one-to-one.

EXERCISE:-

Find all one-to-one functions from X = {a,b} to Y = {u,v}

SOLUTION:-

There are two one-to-one functions from X to Y defined by the arrow diagrams.

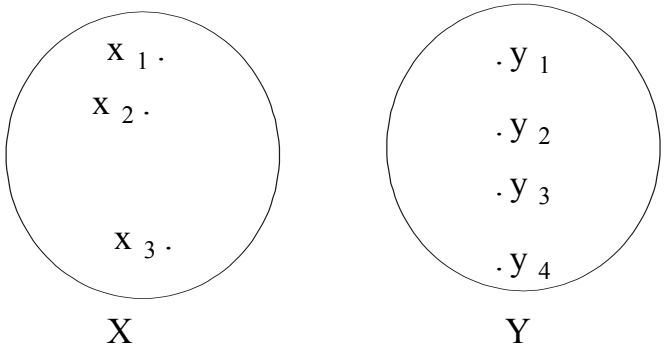


EXERCISE:-

How many one-to-one functions are there from a set with three elements to a set with four elements.

SOLUTION:-

Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3, y_4\}$



x_1 may be mapped to any of the 4 elements of Y. Then x_2 may be mapped to any of the remaining 3 elements of Y & finally x_3 may be mapped to any of the remaining 2 elements of Y.

Hence, total no. of one-to-one functions from X to Y are

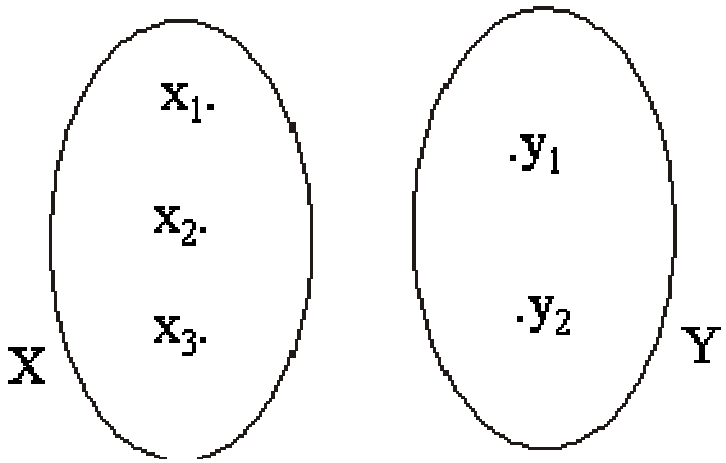
$4 \times 3 \times 2 = 24$

EXERCISE:-

How many one-to-one functions are there from a set with three elements to a set with two elements.

SOLUTION:-

Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$

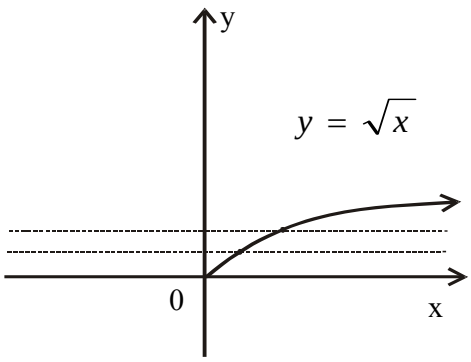


Two elements in X could be mapped to the two elements in Y separately. But there is no new element in Y to which the third element in X could be mapped. Accordingly there is no one-to-one function from a set with three elements to a set with two elements.

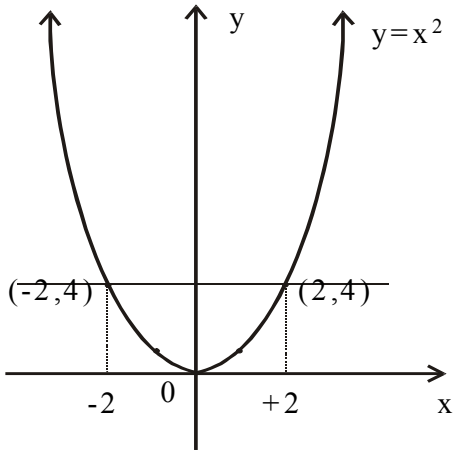
GRAPH OF ONE-TO-ONE FUNCTION

A graph of a function f is one-to-one iff every horizontal line intersects the graph in at most one point.

EXAMPLE:-



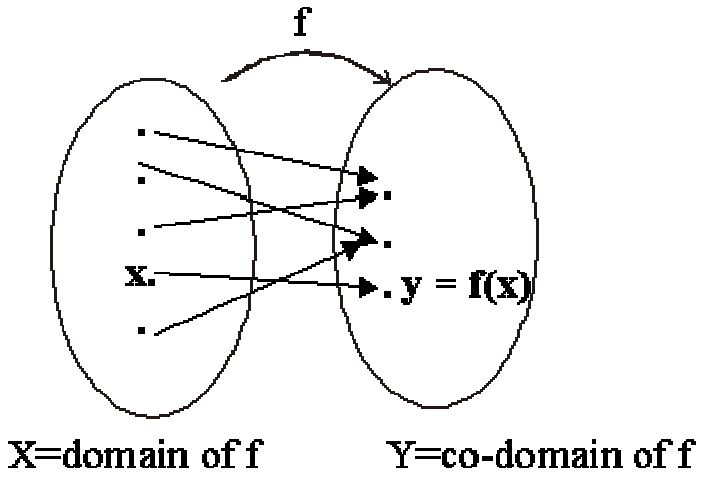
ONE-TO-ONE FUNCTION
from R^+ to R



NOT ONE-TO-ONE FUNCTION
From R to R^+

SURJECTIVE FUNCTION or ONTO FUNCTION

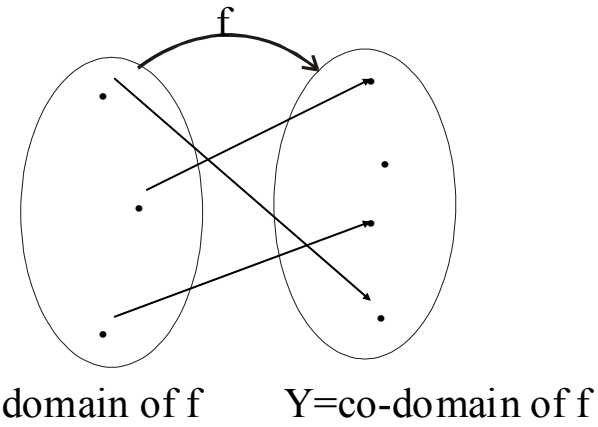
Let f: X→Y be a function. f is surjective or onto if, and only if,"∀ y ε Y, ∃ x εX such that f(x) = y. That is, f is onto if every element of its co-domain is the image of some element(s) of its domain.i.e., co-domain of f = range of f



Each element y in Y equals f(x) for at least one x in X

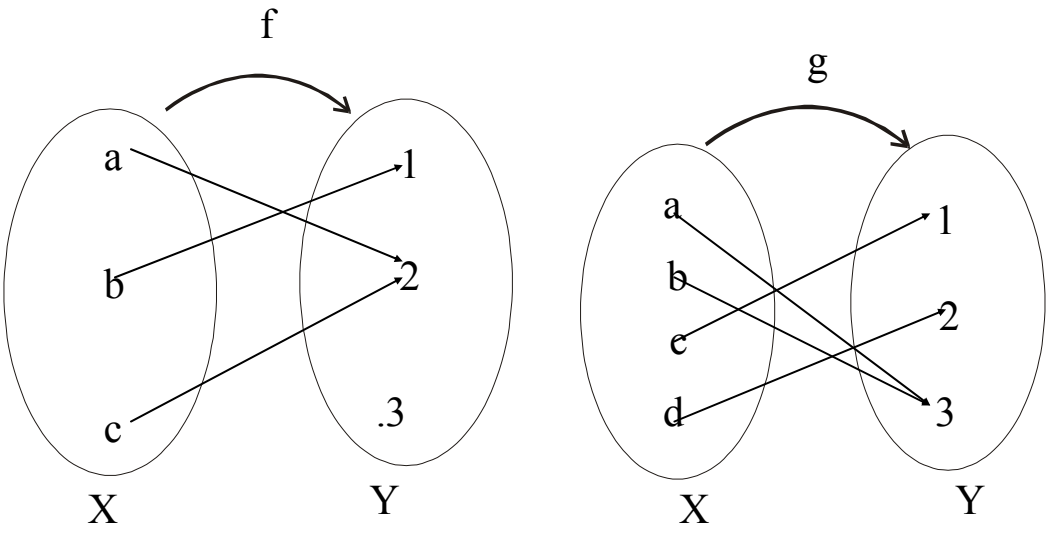
FUNCTION NOT ONTO

A function f:X→Y is not onto iff there exists yε Y such that ∀x εX, f(x) ≠y.
That is, there is some element in Y that is not the image of any element in X.



EXAMPLE:-

Which of the arrow diagrams define onto functions?



SOLUTION:-

f is not onto because 3 ≠ f(x) for any x in X. g is clearly onto because each element of Y equals g(x) for some x in X.

as 1 = g(c);,2 = g(d);3 = g(a) = g(b)

EXAMPLE:-

Define f: R →R by the rule

f(x) = 4x-1 for all x ∈R

Is f onto? Prove or give a counter example.

SOLUTION:-

Let y ∈R. We search for an x ∈ R such that

f(x) = y

or 4x-1 = y (by definition of f)

Solving it for x, we find x=y+1 $x = \frac{y+1}{4} \in R$ Hence for every y ∈R, there

exists $x = \frac{y+1}{4} \in R$ such that

$$f(x) = f\left(\frac{y+1}{4}\right)$$

$$= 4.\left(\frac{y+1}{4}\right) - 1 = (y+1) - 1 = y$$

Hence f is onto.

EXAMPLE:-

Define h: Z →Z by the rule

h(n) = 4n - 1 for all n ∈ Z

Is h onto? Prove or give a counter example.

SOLUTION:-

Let m ∈Z. We search for an n ∈Z such that h(n) = m.

or 4n - 1 = m (by definition of h)

Solving it for n, we find $n = \frac{m+1}{4}$

But $n = \frac{m+1}{4}$ is not always an integer for all m ∈Z.

As a **counter example**, let m = 0∈ Z, then

h(n) = 0

⇒ 4n-1 = 0

⇒ 4n = 1

⇒ $n = \frac{1}{4} \notin Z$

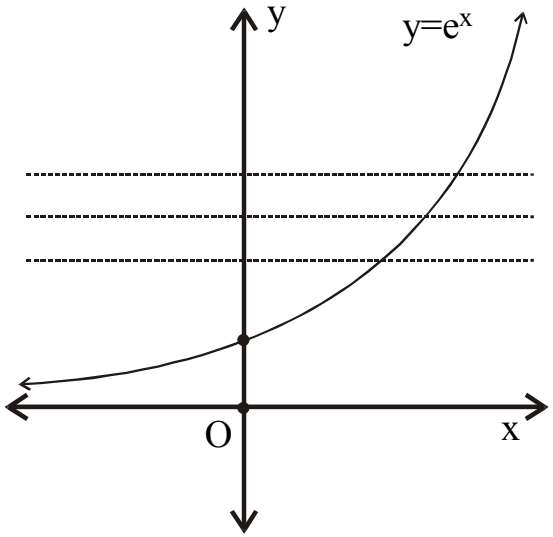
Hence there is no integer n for which h(n) = 0.

Accordingly, h is not onto.

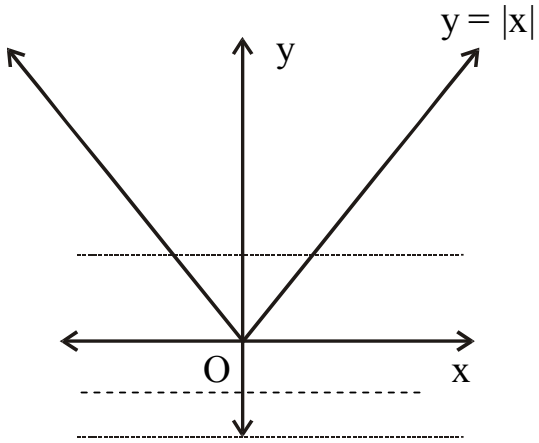
GRAPH OF ONTO FUNCTION

A graph of a function f is onto iff every horizontal line intersects the graph in at least one point.

EXAMPLE:-



ONTO FUNCTION from R to R^+



NOT ONTO FUNCTION FROM R to R

EXERCISE:-

Let X = {1,5,9} and Y = {3,4,7}.Define g: X →Y by specifying that

g(1) = 7, g(5) = 3, g(9) = 4

Is g one-to-one? Is g onto?

SOLUTION:-

g is one-to-one because each of the three elements of X are mapped to a different elements of Y by g.

$g(1) \neq g(5)$, $g(1) \neq g(a)$, $g(5) \neq g(a)$

g is onto as well, because each of the three elements of co-domain Y of g is the image of some element of the domain of g.

3 = g(5), 4 = g(9), 7 = g(1)

EXERCISE:-

Define f: P({a,b,c})→Z as follows:

for all A∈P ({a,b,c}), f(A)= the number of elements in A.

a. Is f one-to-one? Justify.

b. Is f onto? Justify.

SOLUTION:-

a. f is not one-to-one because f({a}) = 1 and f({b}) = 1 but {a}≠ {b}

b. f is not onto because, there is no element of P({a,b,c}) that is mapped to 4 ∈Z.

EXERCISE:-

Determine if each of the functions is injective or surjective.

a. f: $Z \rightarrow Z^+$ define as f(x) = |x|

b. g: $Z^+ \rightarrow Z^+ \times Z^+$ defined as g(x) = (x,x+1)

SOLUTION:-

a. **f is not injective**, because

f(1) = |1| = 1 and f(-1) = |-1| = 1

i.e., f(1) = f(-1) but $1 \neq -1$

f is onto, because for every a$\in Z^+$, there exist –a and +a in Z such that

f(-a) = |-a| = a and f(a) = |a| = a

b. g: $Z^+ \rightarrow Z^+ \times Z^+$ defined as g(x) = (x,x+1)

Let $g(x_1) = g(x_2)$ for $x_1, x_2 \in Z^+$

$\Rightarrow$ $(x_1, x_1 +1) = (x_2, x_2+1)$(by definition of g)

$\Rightarrow$ $x_1 = x_2$ and $x_1 + 1 = x_2 + 1$

(by equality of ordered pairs)

$\Rightarrow$ $x_1 = x_2$

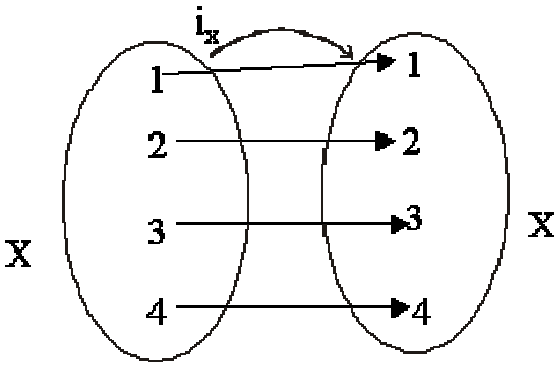
Thus if $g(x_1) = g(x_2)$ then $x_1 = x_2$

Hence **g is one-to-one**.

g is not onto because (1,1) $\in Z^+ \times Z^+$ is not the image of any element of Z^+.

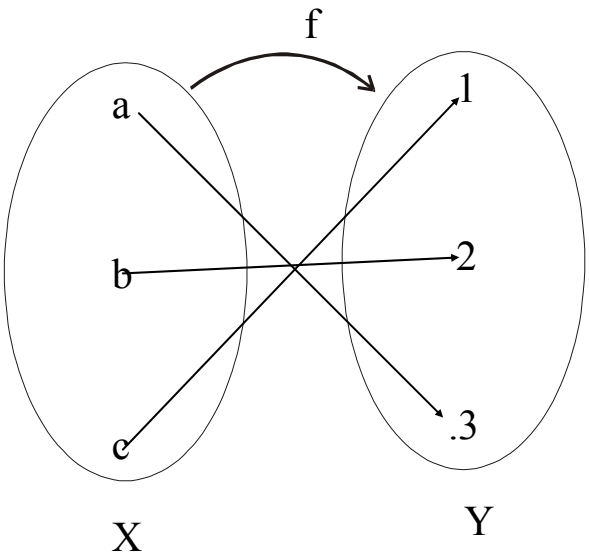
BIJECTIVE FUNCTION or ONE-TO-ONE CORRESPONDENCE

A function f: X→Y that is both one-to-one (injective) and onto (surjective) is called a bijective function or a one-to-one correspondence.



EXAMPLE:-

The function f: X→Y defined by the arrow diagram is both one-to-one and onto; hence a bijective function.



EXERCISE:-

Let f: R $\rightarrow$R be defined by the rule $f(x) = x^3$.Show that f is a bijective.

SOLUTION:-

f is one-to-one

Let $f(x_1) = f(x_2)$ for $x_1, x_2 \in R$

$\Rightarrow$ $x_1^3 = x_2^3$

$\Rightarrow$ $x_1^3 - x_2^3 = 0$

$\Rightarrow$ $(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$

$\Rightarrow$ $x_1 - x_2 = 0$ or $x_1^2 + x_1x_2 + x_2^2 = 0$

$\Rightarrow$ $x_1 = x_2$ (the second equation gives no real solution)

Accordingly f is one-to-one.

f is onto

Let $y \in R$. We search for a $x \in R$ such that

$f(x)=y$

$\Rightarrow$ $x^3 = y$ (by definition of f)

or $x = (y)^{1/3}$

Hence for $y \in R$, there exists $x = (y)^{1/3} \in R$ such that

$f(x) = f((y)^{1/3})$

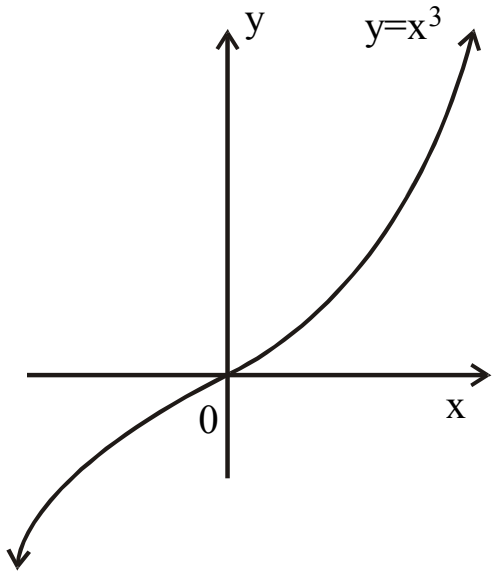
$= ((y)^{1/3})^3 = y$

Accordingly f is onto.

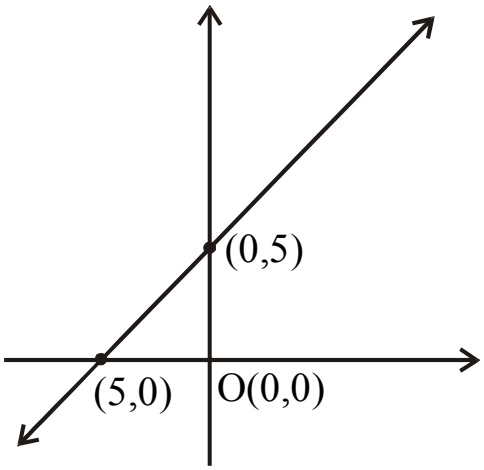
Thus, f is a bijective.

GRAPH OF BIJECTIVE FUNCTION

A graph of a function f is bijective iff every horizontal line intersects the graph at exactly one point.



BIJECTIVE FUNCTION
from R to R



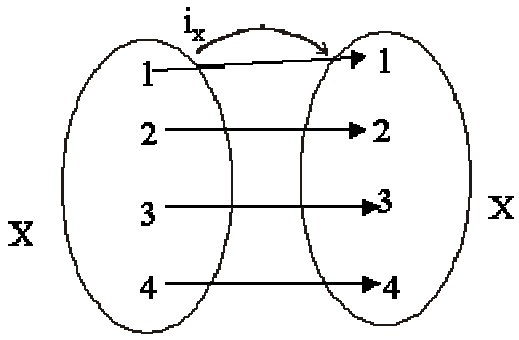
BIJECTIVE FUNCTION
from R to R

IDENTITY FUNCTION ON A SET

Given a set X, define a function i_X from X to X by $i_X(x) = x$ from all $x \in X$.
The function i_X is called the identity function on X because it sends each element of X to itself.

EXAMPLE:-

Let X = {1,2,3,4}. The identity function i_X on X is represented by the arrow diagram



EXERCISE:-

Let X be a non-empty set. Prove that the identity function on X is bijective.

SOLUTION:-

Let $i_X: X \to X$ be the identity function defined as $i_X(x) = x \ \forall \in X$

1. **i_X is injective (one-to-one)**

   Let $i_X(x_1) = i_X(x_2)$ for $x_1, x_2 \in X$

   $\Rightarrow \ x_1 = x_2$ (by definition of i_X)

   Hence i_X is one-to-one.

2. **i_X is surjective (onto)**

   Let $y \in X$ (co-domain of i_X) Then there exists $y \in X$ (domain of i_X) such that $i_X(y) = y$

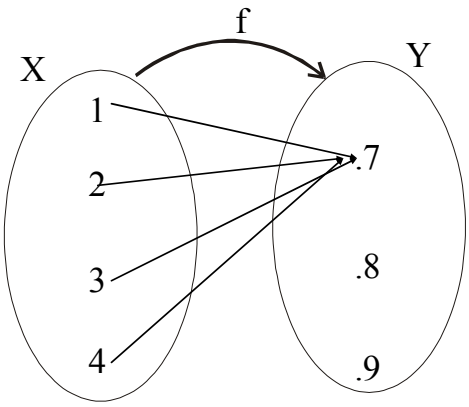
   Hence i_X is onto. Thus, i_X being injective and surjective is bijective.

CONSTANT FUNCTION

A function f:X→Y is a constant function if it maps (sends) all elements of X to one element of Y i.e.
$\forall x \in X, f(x) = c,$ for some $c \in Y$

EXAMPLE:-

The function f defined by the arrow diagram is constant.



REMARK:-

1. A constant function is one-to-one iff its domain is a singleton.
2. A constant function is onto iff its co-domain is a singleton.

LECTURE # 13

Sequences

SEQUENCE

A sequence is just a list of elements usually written in a row.

EXAMPLES:-

1. 1, 2, 3, 4, 5, …
2. 4, 8, 12, 16, 20,…
3. 2, 4, 8, 16, 32, …
4. 1, 1/2, 1/3, 1/4, 1/5, …
5. 1, 4, 9, 16, 25, …
6. 1, -1, 1, -1, 1, -1, …

NOTE:-

The symbol "…" is called ellipsis, and reads "and so forth"

FORMAL DEFINITION

A sequence is a function whose domain is the set of integers greater than or equal to a particular integer n_0.

Usually this set is the set of Natural numbers {1, 2, 3, …} or the set of whole numbers {0, 1, 2, 3, …}.

NOTATION:-

We use the notation a_n to denote the image of the integer n, and call it a term of the sequence. Thus

$a_1, a_2, a_3, a_4 \ldots, a_n, \ldots$

represent the terms of a sequence defined on the set of natural numbers N.

Note that a sequence is described by listing the terms of the sequence in order of increasing subscripts.

FINDING TERMS OF A SEQUENCE GIVEN BY AN EXPLICIT FORMULA

An explicit formula or general formula for a sequence is a rule that shows how the values of a_k depends on k.

EXAMPLE:-

Define a sequence $a_1, a_2, a_3, \ldots$ by the explicit formula

$$a_k = \frac{k}{k+1} \qquad \text{for all integers } k \geq 1$$

The first four terms of the sequence are:

$$a_1 = \frac{1}{1+1} = \frac{1}{2}, a_2 = \frac{2}{2+1} = \frac{2}{3}, a_3 = \frac{3}{3+1} = \frac{3}{4}$$

$$\textit{and} \quad \textit{fourth term is } a_4 = \frac{4}{4+1} = \frac{4}{5}$$

EXAMPLE:-

Write the first four terms of the sequence defined by the formula

$b_j = 1 + 2^j$, for all integers $j \geq 0$

SOLUTION:-

$b_0 = 1 + 2^0 = 1 + 1 = 2$

$b_1 = 1 + 2^1 = 1 + 2 = 3$

$b_2 = 1 + 2^2 = 1 + 4 = 5$

$b_3 = 1 + 2^3 = 1 + 8 = 9$

REMARK:-

The formula $b_j = 1 + 2^j$, for all integers $j \geq 0$ defines an infinite sequence having infinite number of values.

EXERCISE:-

Compute the first six terms of the sequence defined by the formula $C_n = 1 + (-1)^n$ for all integers $n \geq 0$

SOLUTION :-

$C_0 = 1 + (-1)^0 = 1 + 1 = 2$ $C_1 = 1 + (-1)^1 = 1 + (-1) = 0$

$C_2 = 1 + (-1)^2 = 1 + 1 = 2$ $C_3 = 1 + (-1)^3 = 1 + (-1) = 0$

$C_4 = 1 + (-1)^4 = 1 + 1 = 2$ $C_5 = 1 + (-1)^5 = 1 + (-1) = 0$

REMARK:-

(1)If n is even, then $C_n = 2$ and if n is odd, then $C_n = 0$

Hence, the sequence oscillates endlessly between 2 and 0.

(2)An infinite sequence may have only a finite number of values.

EXAMPLE:-

Write the first four terms of the sequence defined by

$$C_n = \frac{(-1)^n n}{n+1} \quad \text{for all integers } n \geq 1$$

SOLUTION:-

$$C_1 = \frac{(-1)^1(1)}{1+1} = \frac{-1}{2}, C_2 = \frac{(-1)^2(2)}{2+1} = \frac{2}{3}, C_3 = \frac{(-1)^3(3)}{3+1} = \frac{-3}{4}$$

$$And \quad fourth \ term \ is C_4 = \frac{(-1)^4(4)}{4+1} = \frac{4}{5}$$

REMARK:A sequence whose terms alternate in sign is called an alternating sequence.

EXERCISE:-

Find explicit formulas for sequences with the initial terms given:

1. 0, 1, -2, 3, -4, 5, …

SOLUTION:-

$a_n = (-1)^{n+1} n$ for all integers $n \geq 0$

2. $1 - \frac{1}{2}, \frac{1}{2} - \frac{1}{3}, \frac{1}{3} - \frac{1}{4}, \frac{1}{4} - \frac{1}{5}, \ldots$

SOLUTION:-

$$b_k = \frac{1}{k} - \frac{1}{k+1} \quad \text{for all integers } n \geq 1$$

3. 2, 6, 12, 20, 30, 42, 56, …

SOLUTION:-

$C_n = n (n + 1)$ for all integers $n \geq 1$

4. 1/4, 2/9, 3/16, 4/25, 5/36, 6/49, …

SOLUTION:-

$$d_i = \frac{i}{(i+1)^2} \quad \text{for all integers} \quad i \geq 1$$

OR

$$d_j = \frac{j+1}{(j+2)^2} \quad \text{for all integers} \quad j \geq 0$$

ARITHMETIC SEQUENCE

A sequence in which every term after the first is obtained from the preceding term by adding a constant number is called an arithmetic sequence or arithmetic progression (A.P.)

The constant number, being the difference of any two consecutive terms is called the common difference of A.P., commonly denoted by "d".

EXAMPLES:-

1. 5, 9, 13, 17, … (common difference = 4)

2. 0, -5, -10, -15, … (common difference = -5)
3. x + a, x + 3a, x + 5a, … (common difference = 2a)

GENERAL TERM OF AN ARITHMETIC SEQUENCE

Let **a** be the first term and **d** be the common difference of an arithmetic sequence. Then the sequence is a, a+d, a+2d, a+3d, …

If a_i, for $i \geq 1$, represents the terms of the sequence then

a_1 = first term = a = a + (1-1) d

a_2 = second term = a + d = a + (2-1) d

a_3 = third term = a + 2d = a + (3 -1) d

By symmetry

a_n = nth term = a + (n - 1)d for all integers $n \geq 1$.

EXAMPLE:-

Find the 20th term of the arithmetic sequence

3, 9, 15, 21, …

SOLUTION:-

Here a = first term = 3

d = common difference = 9 - 3 = 6

n = term number = 20

a_{20} = value of 20th term = ?

Since $a_n = a + (n - 1)\, d$; $n \geq 1$

$\therefore \quad a_{20} = 3 + (20 - 1)\, 6$

$= 3 + 114$

$= 117$

EXAMPLE:-

Which term of the arithmetic sequence

4, 1, -2, …, is -77

SOLUTION:-

Here a = first term = 4

d = common difference = 1 - 4 = -3

a_n = value of nth term = - 77

n = term number = ?

Since

$an = a + (n - 1)\, d$ $n \geq 1$

$\Rightarrow \quad -77 = 4 + (n - 1)(-3)$

$\Rightarrow \quad -77 - 4 = (n - 1)(-3)$

OR

$$\frac{-81}{-3} = n - 1$$

OR

$27 = n - 1$

$n = 28$

Hence –77 is the 28th term of the given sequence.

EXERCISE:-

Find the 36th term of the arithmetic sequence whose 3rd term is 7 and 8th term is 17.

SOLUTION:-

Let **a** be the first term and **d** be the common difference of the arithmetic sequence.

Then

$a_n = a + (n - 1)d \quad n \geq 1$

$\Rightarrow \quad a_3 = a + (3 - 1)\, d$

and $a_8 = a + (8 - 1)\, d$

Given that $a_3 = 7$ and $a_8 = 17$. Therefore

$7 = a + 2d$……(1)

and $17 = a + 7d$……..(2)

Subtracting (1) from (2), we get,

$10 = 5d$

$\Rightarrow \quad d = 2$

Substituting d = 2 in (1) we have

$7 = a + 2(2)$

which gives a = 3

Thus, $a_n = a + (n - 1)\, d$

$a_n = 3 + (n - 1)\, 2$ (using values of a and d)

Hence the value of 36th term is

$a_{36} = 3 + (36 - 1)\, 2$

$= 3 + 70$

$= 73$

GEOMETRIC SEQUENCE

A sequence in which every term after the first is obtained from the preceding term by multiplying it with a constant number is called a geometric sequence or geometric progression (G.P.)

The constant number, being the ratio of any two consecutive terms is called the common ratio of the G.P. commonly denoted by "r".

EXAMPLE:-

1. 1, 2, 4, 8, 16, … (common ratio = 2)
2. 3, - 3/2, 3/4, - 3/8, … (common ratio = - 1/2)
3. 0.1, 0.01, 0.001, 0.0001, … (common ratio = 0.1 = 1/10)

GENERAL TERM OF A GEOMETRIC SEQUENCE:

Let **a** be the first tem and **r** be the common ratio of a geometric sequence. Then the sequence is a, ar, ar^2, ar^3, …

If a_i, for $i \geq 1$ represent the terms of the sequence, then

a_1 = first term = $a = ar^{1-1}$

a_2 = second term = $ar = ar^{2-1}$

a_3 = third term = $ar^2 = ar^{3-1}$

……

……

a_n = nth term = ar^{n-1}; for all integers $n \geq 1$

EXAMPLE:-

Find the 8th term of the following geometric sequence

4, 12, 36, 108, …

SOLUTION:-

Here a = first term = 4

r = common ratio = $\frac{12}{4} = 3$

n = term number = 8

a_8 = value of 8th term = ?

Since $a_n = ar^{n-1}$; $n \geq 1$

$\Rightarrow \quad a_8 = (4)(3)^{8-1}$

$= 4\,(2187)$

$= 8748$

EXAMPLE:-

Which term of the geometric sequence is 1/8 if the first term is 4 and common ratio ½

SOLUTION:-

Given a = first term = 4

r = common ratio = 1/2

a_n = value of the nth term = 1/8

n = term number = ?

Since $a_n = ar^{n-1}$ $n \geq 1$

$$\Rightarrow \quad \frac{1}{8} = 4\left(\frac{1}{2}\right)^{n-1}$$

$$\Rightarrow \quad \frac{1}{32} = \left(\frac{1}{2}\right)^{n-1}$$

$$\Rightarrow \quad \left(\frac{1}{2}\right)^{5} = \left(\frac{1}{2}\right)^{n-1}$$

$$\Rightarrow \quad n-1=5 \qquad \Rightarrow n=6$$

Hence 1/8 is the 6th term of the given G.P.

EXAMPLE:-

Write the geometric sequence with positive terms whose second term is 9 and fourth term is 1.

SOLUTION:-

Let **a** be the first term and **r** be the common ratio of the geometric sequence. Then

$a_n = ar^{n-1}$ $\quad n \geq 1$

Now $a_2 = ar^{2-1}$

$\Rightarrow$ $9 = ar$…….(1)

Also $a_4 = ar^{4-1}$

$1 = ar^3$ ……(2)

Dividing (2) by (1), we get,

$$\frac{1}{9} = \frac{ar^3}{ar}$$

$$\Rightarrow \quad \frac{1}{9} = r^2$$

$$\Rightarrow \quad r = \frac{1}{3} \qquad \left(\text{rejecting } r = -\frac{1}{3}\right)$$

Substituting r = 1/3 in (1), we get

$$9 = a\left(\frac{1}{3}\right)$$

$$\Rightarrow \quad a = 9 \times 3 = 27.$$

Hence the geometric sequence is

27, 9, 3, 1, 1/3, 1/9, …

SEQUENCES IN COMPUTER PROGRAMMING

An important data type in computer programming consists of finite sequences known as one-dimensional arrays; a single variable in which a sequence of variables may be stored.

EXAMPLE:-

The names of k students in a class may be represented by an array of k elements "name" as:

name [0], name[1], name[2], …, name[k-1]

LECTURE # 14

Series

SERIES

The sum of the terms of a sequence forms a series. If $a_1, a_2, a_3, \ldots$ represent a sequence of numbers, then the corresponding series is

$$a_1 + a_2 + a_3 + \ldots = \sum_{k=1}^{\infty} a_k$$

SUMMATION NOTATION

The capital Greek letter sigma $\sum$ is used to write a sum in a short hand notation.
where k varies from 1 to n represents the sum given in expanded form by
$= a_1 + a_2 + a_3 + \ldots + a_n$
More generally if m and n are integers and $m \le n$, then the summation from k equal m to n of a_k is

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

Here **k** is called the index of the summation; **m** the lower limit of the summation and **n** the upper limit of the summation.

COMPUTING SUMMATIONS

Let $a_0 = 2$, $a_1 = 3$, $a_2 = -2$, $a_3 = 1$ and $a_4 = 0$.Compute each of the summations:

1. $\sum_{i=0}^{4} a_i$ 2. $\sum_{j=0}^{2} a_{2j}$ $\sum_{k=1}^{1} a_k$

SOLUTION:-

$$1.\ \sum_{i=0}^{4} a_i = a_0 + a_1 + a_2 + a_3 + a_4 = 2 + 3 + (-2) + 1 + 0 = 4$$

$$2 \sum_{j=0}^{2} a_{2j} = a_0 + a_1 + a_2 = 2 - 2 = 0$$

EXAMPLE:-

Compute the summations

$$1.\ \sum_{i=1}^{3} (2i-1) = [2(1)-1]+[2(2)-1]+[2(3)-1]$$
$$= 1+3+5$$
$$= 9$$

$$2.\ \sum_{k=-1}^{1} (k^3+2) = [(-1)^3+2]+[(0)^3+2]+[(1)^3+2]$$
$$= [-1+2]+[0+2]+[1+2]$$
$$= 1+2+3$$
$$= 6$$

SUMMATION NOTATION TO EXPANDED FORM

Write the summation to expanded form: $\sum_{i=0}^{n} \frac{(-1)^i}{i+1}$

SOLUTION:-

$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1} = \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} + \frac{(-1)^3}{3+1} + \cdots + \frac{(-1)^n}{n+1}$$

$$= \frac{1}{1} + \frac{(-1)}{2} + \frac{1}{3} + \frac{(-1)}{4} + \cdots + \frac{(-1)^n}{n+1}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1}$$

EXPANDED FORM TO SUMMATION NOTATION

Write the following using summation notation:

1. $\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}$

SOLUTION:-

We find the kth term of the series.

The numerators forms an arithmetic sequence 1, 2, 3,…, n+1, in which

a = first term = 1

& d = common difference = 1

$a_k = a + (k - 1)d$

$= 1 + (k - 1)(1) = 1 + k - 1 = k$

Similarly, the denominators forms an arithmetic sequence

n, n+1, n+2, …, 2n, in which

a = first term = n

d = common difference = 1

∴ $a_k = a + (k - 1)\,d$

$= n + (k - 1)(1)$

$= k + n - 1$

Hence the kth term of the series is

$$\frac{k}{(n-1)+k}$$

And the expression for the series is given by

$$\therefore \quad \frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n} = \sum_{k=1}^{n+1} \frac{k}{(n-1)+k}$$

$$= \sum_{k=0}^{n} \frac{k+1}{n+k}$$

TRANSFORMING A SUM BY A CHANGE OF VARIABLE

Consider $\sum_{k=1}^{3} k^2 = 1^2 + 2^2 + 3^2$

and $\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2$

Hence $\sum_{k=1}^{3} k^2 = \sum_{i=1}^{3} i^2$

The index of a summation can be replaced by any other symbol. The index of a summation is therefore called a dummy variable.

EXERCISE:-

Consider $\sum_{k=1}^{n+1} \frac{k}{(n-1)+k}$

Substituting k = j + 1 so that j = k – 1
When k = 1, j = k - 1 = 1 - 1 = 0
When k = n + 1, j = k - 1 = (n + 1) - 1 = n
Hence

$$\sum_{k=1}^{n+1} \frac{k}{(n-1)+k} = \sum_{j=0}^{n} \frac{j+1}{(n-1)+(j+1)}$$

$$= \sum_{j=0}^{n} \frac{j+1}{n+j} = \sum_{k=0}^{n} \frac{k+1}{n+k} \quad \text{(changing variable)}$$

Transform by making the change of variable j = i - 1, in the summation

$$\sum_{i=1}^{n-1} \frac{i}{(n-i)^2} \quad **$$

PROPERTIES OF SUMMATIONS

1. $\sum_{k=m}^{n} (a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k; \quad a_k, b_k \in R$
2. $\sum_{k=m}^{n} ca_k = c\sum_{k=m}^{n} a_k \quad c \in R$
3. $\sum_{k=a-i}^{b-i} (k+i) = \sum_{k=a}^{b} k \quad i \in N$
4. $\sum_{k=a+i}^{b+i} (k-i) = \sum_{k=a}^{b} k \quad i \in N$
5. $\sum_{k=1}^{n} c = c + c + \cdots + c = nc$

EXERCISE:-

Express the following summation more simply:

$$3\sum_{k=1}^{n} (2k-3) + \sum_{k=1}^{n} (4-5k)$$

SOLUTION:-

$$3\sum_{k=1}^{n}(2k-3)+\sum_{k=1}^{n}(4-5k)$$

$$=3\sum_{k=1}^{n}3(2k-3)+\sum_{k=1}^{n}(4-5k)$$

$$=\sum_{k=1}^{n}[3(2k-3)+(4-5k)]$$

$$=\sum_{k=1}^{n}(k-5)$$

$$=\sum_{k=1}^{n}k-\sum_{k=1}^{n}5$$

$$=\sum_{k=1}^{n}k-5n$$

ARITHMETIC SERIES

The sum of the terms of an arithmetic sequence forms an arithmetic series (A.S). For example

$$1 + 3 + 5 + 7 + \ldots$$

is an arithmetic series of positive odd integers.

In general, if a is the first term and d the common difference of an arithmetic series, then the series is given as: $a + (a+d) + (a+2d) + \ldots$

SUM OF n TERMS OF AN ARITHMETIC SERIES

Let a be the first term and d be the common difference of an arithmetic series. Then its nth term is:

$$a_n = a + (n - 1)d; \qquad n \geq 1$$

If S_n denotes the sum of first n terms of the A.S, then

$$S_n = a + (a + d) + (a + 2d) + \ldots + [a + (n-1)\, d]$$
$$= a + (a+d) + (a + 2d) + \ldots + a_n$$
$$= a + (a+d) + (a + 2d) + \ldots + (a_n - d) + a_n \ldots\ldots\ldots(1)$$

where $a_n = a + (n - 1)\, d$

Rewriting the terms in the series in reverse order,

$$S_n = a_n + (a_n - d) + (a_n - 2d) + \ldots + (a + d) + a \ldots\ldots\ldots(2)$$

Adding (1) and (2) term by term, gives

$$2S_n = (a + a_n) + (a + a_n) + (a + a_n) + \ldots + (a + a_n) \qquad (n \text{ terms})$$
$$2S_n = n(a + a_n)$$
$$\Rightarrow \quad S_n = n(a + a_n)/2$$
$$S_n = n(a + l)/2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Where $l = a_n = a + (n - 1)d$

Therefore

$$S_n = n/2\,[a + a + (n - 1)\, d]$$
$$S_n\ n/2\,[2a + (n - 1)\, d] \ldots\ldots\ldots(4)$$

EXERCISE:-

Find the sum of first n natural numbers.

SOLUTION:-

Let $S_n = 1 + 2 + 3 + \ldots + n$

Clearly the right hand side forms an arithmetic series with

$a = 1, \quad d = 2 - 1 = 1 \qquad$ and $\quad n = n$

$$\therefore \quad S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$

$$= \frac{n}{2}\left[2(1)+(n-1)(1)\right]$$

$$= \frac{n}{2}\left[2+n-1\right]$$

$$= \frac{n(n+1)}{2}$$

EXERCISE:-

Find the sum of all two digit positive integers which are neither divisible by 5 nor by 2.

SOLUTION:-

The series to be summed is:

11 + 13 + 17 + 19 + 21 + 23 + 27 + 29 + … + 91 + 93 + 97 + 99

which is not an arithmetic series.

If we make group of four terms we get

(11 + 13 + 17 + 19) + (21 + 23 + 27 + 29) + (31 + 33 + 37 + 39) + … + (91 + 93 + 97 + 99) = 60 + 100 + 140 + … + 380

which now forms an arithmetic series in which

a = 60; d = 100 - 60 = 40 and l = a_n = 380

To find n, we use the formula

a_n = a + (n - 1) d

$\Rightarrow$ 380 = 60 + (n - 1) (40)

$\Rightarrow$ 380 - 60 = (n - 1) (40)

$\Rightarrow$ 320 = (n - 1) (40)

$$\frac{320}{40} = n-1$$

$$8 = n - 1$$

$\Rightarrow$ n = 9

Now

$$S_n = \frac{n}{2}(a+l)$$

$$\therefore \quad S_9 = \frac{9}{2}(60+380) = 1980$$

GEOMETRIC SERIES

The sum of the terms of a geometric sequence forms a geometric series (G.S.). For example

1 + 2 + 4 + 8 + 16 + …

is geometric series.

In general, if **a** is the first term and **r** the common ratio of a geometric series, then the series is given as: $a + ar + ar^2 + ar^3 + \ldots$

SUM OF n TERMS OF A GEOMETRIC SERIES

Let **a** be the first term and **r** be the common ratio of a geometric series. Then its nth term is:

$a_n = ar^{n-1}$; $n \geq 1$

If Sn denotes the sum of first n terms of the G.S. then

$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}$……(1)

Multiplying both sides by r we get.

$r S_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$……(2)

Subtracting (2) from (1) we get

$S_n - rS_n = a - ar^n$

$$\Rightarrow \quad (1 - r)\, S_n = a\,(1 - r^n)$$

$$\Rightarrow \quad S_n = \frac{a(1-r^n)}{1-r} \qquad (r \neq 1)$$

EXERCISE:-

Find the sum of the geometric series

$$6 - 2 + \frac{2}{3} - \frac{2}{9} + \cdots + \text{ to 10 terms}$$

SOLUTION:-

In the given geometric series

$$a = 6, \quad r = \frac{-2}{6} = -\frac{1}{3} \quad \text{and } n = 10$$

$$\therefore \quad S_n = \frac{a(1-r^n)}{1-r}$$

$$S_{10} = \frac{6\left(1-\left(-\frac{1}{3}\right)^{10}\right)}{1-\left(-\frac{1}{3}\right)} = \frac{6\left(1+\frac{1}{3^{10}}\right)}{\left(\frac{4}{3}\right)}$$

$$= \frac{9\left(1+\frac{1}{3^{10}}\right)}{2}$$

INFINITE GEOMETRIC SERIES

Consider the infinite geometric series

$$a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$$

then

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r} \qquad (r \neq 1)$$

If $S_n \to S$ as $n \to \infty$, then the series is convergent and S is its sum.

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$

$$\therefore \quad S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

$$= \frac{a}{1-r}$$

If S_n increases indefinitely as n becomes very large then the series is said to be divergent.

EXERCISE:-

Find the sum of the infinite geometric series:

$$\frac{9}{4} + \frac{3}{2} + 1 + \frac{2}{3} + \cdots$$

SOLUTION:-

Here we have

$$a = \frac{9}{4}, \quad r = \frac{3/2}{9/4} = \frac{2}{3}$$

Note that $|r| < 1$ So we can use the above formula.

$$\therefore \quad S = \frac{a}{1-r}$$

$$= \frac{9/4}{1-2/3}$$

$$= \frac{9/4}{1/3} = \frac{9}{4} \times \frac{3}{1} = \frac{27}{4}$$

EXERCISE:-

Find a common fraction for the recurring decimal 0.81

SOLUTION:-

$$0.81 = 0.8181818181 \ldots$$

$$= 0.81 + 0.0081 + 0.000081 + \ldots$$

which is an infinite geometric series with

$$a = 0.81, \quad r = \frac{0.0081}{0.81} = 0.01$$

$$\therefore \quad \text{Sum} = \frac{a}{1-r}$$

$$= \frac{0.81}{1-0.01} = \frac{0.81}{0.99}$$

$$= \frac{81}{99} = \frac{9}{11}$$

IMPORTANT SUMS

1. $1+2+3+\cdots+n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

2. $1^2+2^2+3^2+\cdots+n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

3. $1^3+2^3+3^3+\cdots+n^3 = \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)}{4} = \left[\frac{n(n+1)}{2}\right]^2$

EXERCISE:-

Sum to n terms the series $1 \cdot 5 + 5 \cdot 11 + 9 \cdot 17 + \ldots$

SOLUTION:-

1·5+5·11+9·17+…

Here firstly we have to find the ***kth*** term of the series. In fact the above series constructed from to arithmetic sequences;

1) **1, 5, 9,-----**
2) **5, 11, 17,----**

For 1st sequence; **1, 5, 9,-----**

1st term = a =1, d = 5 – 1= 4, so the kth term

a_k = a + (k-1) d = 1 + (k -1) 4 = 4k – 3

for 2nd sequence: **5, 11, 17,----**

1st term = 5, d = 11 – 5 =6 , so the kth term

a_k = a + (k-1) d = 5 + (k -1) 6 = 6k – 1

Now carefully note the pattern;

1st term of the series =

(1^{st} term of the sequence: **1, 5, 9,-----**)(1^{st} term of the seq: **5, 11, 17,----**) = **1·5**
2^{nd} term of the series =
(2^{nd} term of the sequence: **1, 5, 9,-----**)(2^{nd} term of the seq: **5, 11, 17,----**) = **5 ·11**

3^{rd} term of the series =
(3^{rd} term of the sequence: **1, 5, 9,-----**)(3^{rd} term of the seq: **5, 11, 17,----**) = **9·17**

So the kth term of the series =T_K =

(kth term of the sequence: **1, 5, 9,---**)(kth term of the seq: **5, 11, 17,--**) = **(4k – 3)(6k – 1)**
$\mathbf{T_K = (4k - 3)(6k - 1) = 24k^2 - 22k + 3}$
Now the sum of series $\mathbf{= T_1 + T_2 + T_3 + \text{-----} + T_K}$
$\mathbf{= \Sigma\, T_K = \Sigma\,(24k^2 - 22k + 3) = 24\Sigma\, k^2 - 22\,\Sigma\, k + 3\,\Sigma 1}$
$\mathbf{= 24\{k(k+1)(2k+1)/6\} - 22\{k(k+1)/2\} + 3k}$
Which on simplification gives
$\mathbf{= 8k^3 + k^2 - 4k}$

LECTURE # 15
Applications of Basic Mathematics

OBJECTIVES
The objectives of the lecture are to learn about:
- Different course modules
- Basic Arithmetic Operations
- Starting Microsoft (MS) Excel
- Using MS Excel to carry out arithmetic operations

BASIC ARITHMETIC OPERATIONS
Five arithmetic operations provide the foundation for all mathematical operations. These are:
- Addition
- Subtraction
- Multiplication
- Division
- Exponents

Example- Addition
12 + 5 = 17

Example- Subtraction
12 - 5 = 7

Example- Multiplication
12 x 5 = 60

Example- Exponent
(4)^2 = 16
(4)^1/2 = 2
(4)^-1/2 = 1/(4)^1/2 = ½ = 0.5

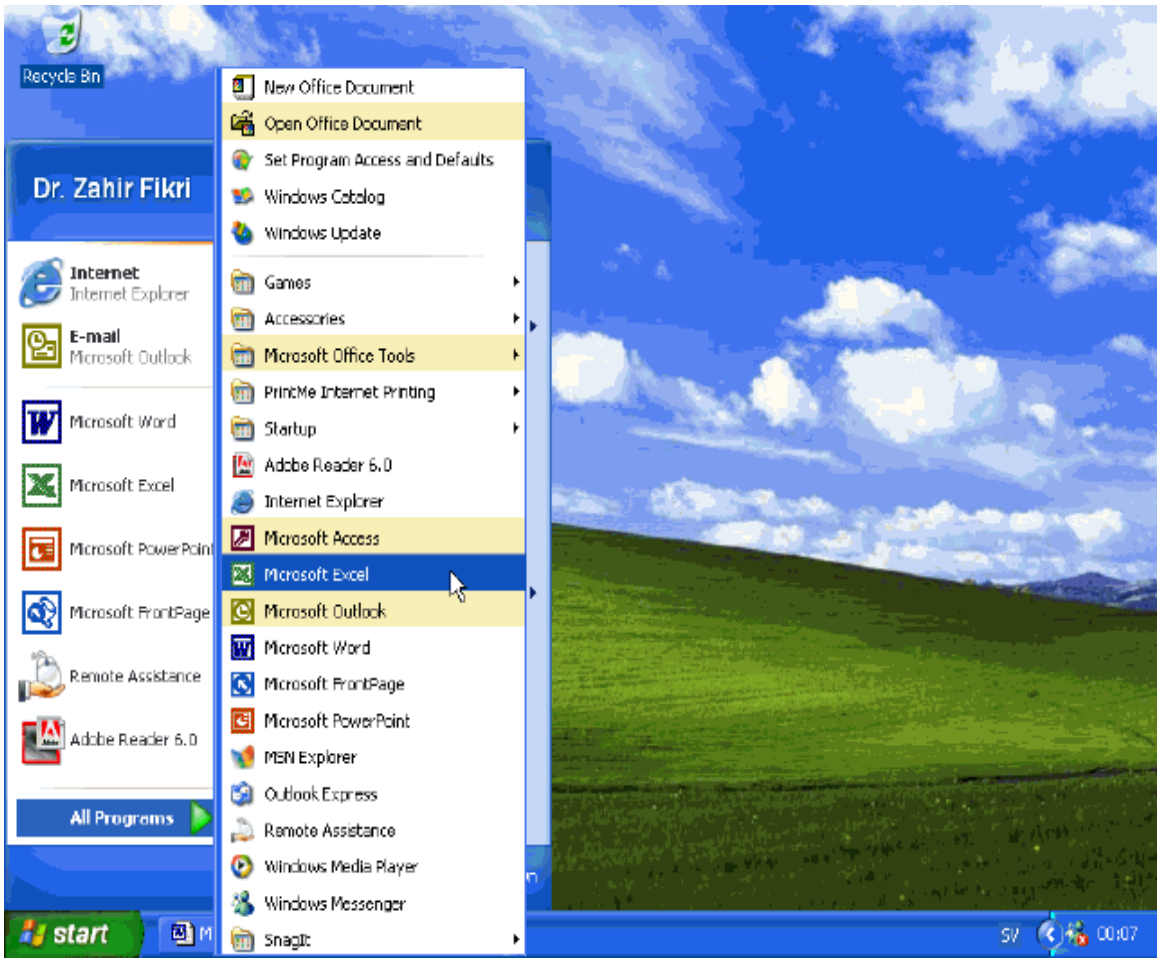
MICROSOFT EXCEL IN BUSINESS MATHEMATICS & STATISTICS

Microsoft Corporation's Spreadsheet software **Excel** is widely used in business mathematics and statistical applications. The latest version of this software is **EXCEL 2002 XP**. This course is based on wide applications of **EXCEL 2002**. It is recommended that you install **EXCEL 2002 XP** software on your computer. If your computer has **Windows 2000** and **EXCEL 2000** even that version of EXCEL can be used as the applications we intend to learn can be done using the earlier version of EXCEL. Those of you who are still working with **Windows 98** and have **EXCEL 97** installed are encouraged to migrate to newer version of EXCEL software.

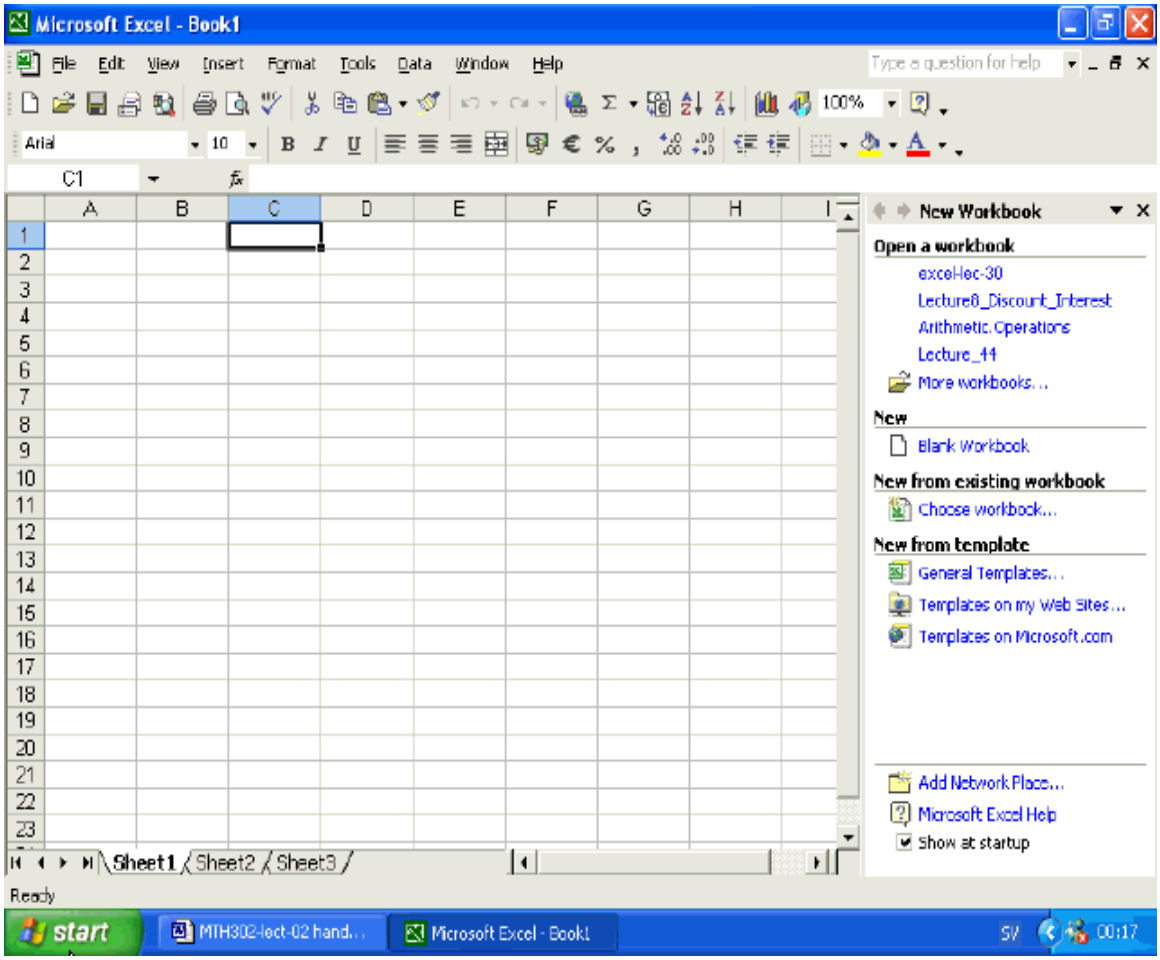
Starting EXCEL 2000 XP

EXCEL 2000 XP can be started by going through the following steps:
Click **Start** on your computer
Click **All Programs**
Click **Microsoft Excel**

The following slides show the operations:



The EXCEL window opens and a blank worksheet becomes available as shown below:



The slide shows a Workbook by the name book1 with three sheets: Sheet1, Sheet2 and Sheet3. The Excel Window has Column numbers starting from A and row numbers starting from 1. the intersection of a row and column is called a **Cell**. The first cell is A1 which is the intersection of column A and row 1. All cells in a Sheet are referenced by a combination of Column name and row number.

Example 1: B15 means cell in column B and row 15.

Example 2: A cell in row 12 and column C has reference C12.

A **Range** defines all cells starting from the leftmost corner where the range starts to the rightmost corner in the last row. The Range is specified by the starting cell, a colon and the ending cell.

Example 3: A **Range** which starts from A1 and ends at D15 is referenced by A1:D15 and has all the cells in columns A to D up to and including row 15.

A value can be entered into a cell by clicking that cell. The mouse pointer which is a rectangle moves to the selected cell. Simply enter the value followed by the **Enter** key. The mouse pointer moves to the cell below.

If you make a mistake while entering the value select the cell again (by clicking it). Enter the new value. The old value is replaced by the new value.

If only one or more digits are to be changed then select the cell. Then double click the mouse. The blinking cursor appears. Either move the arrow key to move to the digit to be changed or move the cursor to the desired position. Enter the new value and delete the undesired value by using the **Del** key.

I suggest that you learn the basic operations of entering, deleting and changing data in a worksheet.

About calculation operators in Excel

In Excel there are four different types of operators:
Arithmetic operators
Comparison operators
Text concatenation operator
Reference operators

The following descriptions are reproduced from Excel's Help file for your ready reference. In the present lecture you are directly concerned with arithmetic operators. However, it is important to learn that the comparison operators are used where calculations are made on the basis of comparisons. The text concatenation operator is used to combine two text strings. The reference operators include ":" and "," or ; as the case maybe. We shall learn the use of these operators in different worksheets. You should look through the Excel Help file to see examples of these functions. Selected material from Excel Help File relating to arithmetic operations is given in in a separate file.
The Excel arithmetic operators are as follows:
Addition. Symbol: + (Example: =5+4 Result: 9)
Subtraction. Symbol: - (Example: =5-4 Result: 1)
Multiplication. Symbol: * (Example: =5*4 Result: 20)
Division. Symbol: / (Example: =12/4 Result: 3)
Percent. Symbol: % (Example: =20% Result: 0.2)
Exponentiation: ^ (Example: =5^2 Result: 25)

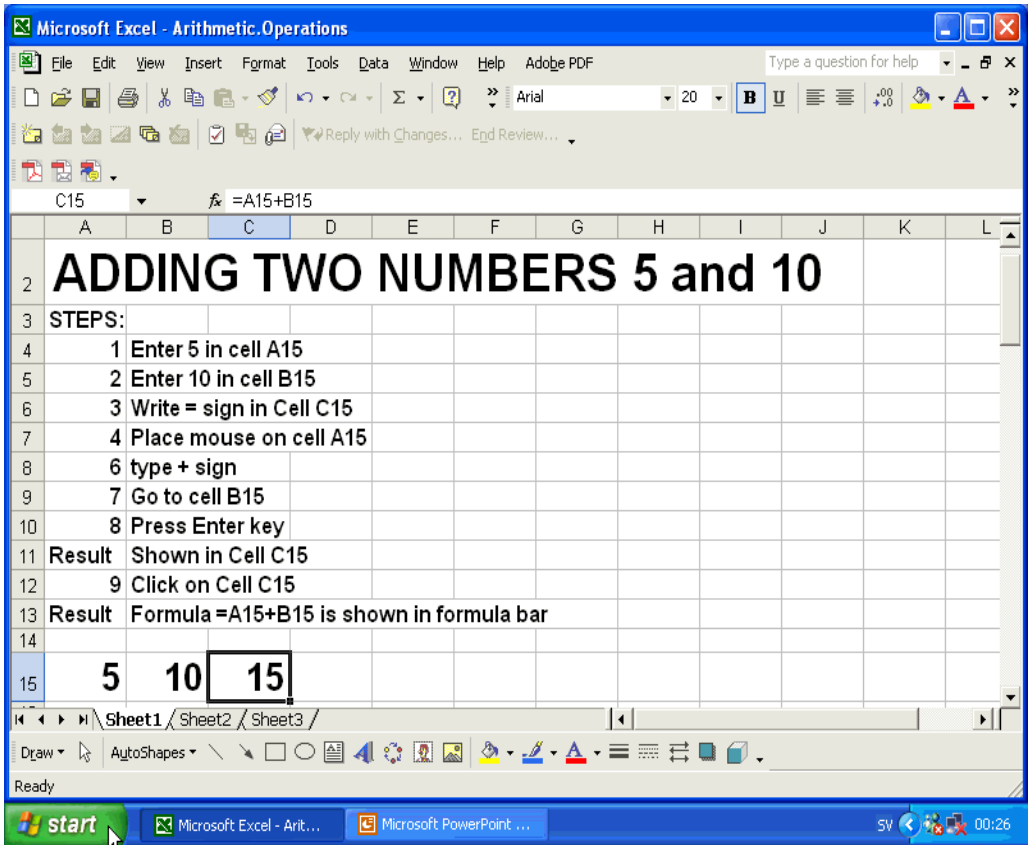
Excel Formulas for Addition

All calculations in Excel are made through formulas which are written in cells where result is required.
Let us do addition of two numbers 5 and 10.
We wish to calculate the addition of two numbers 10 and 5. Let us see how we can add these two numbers in Excel.

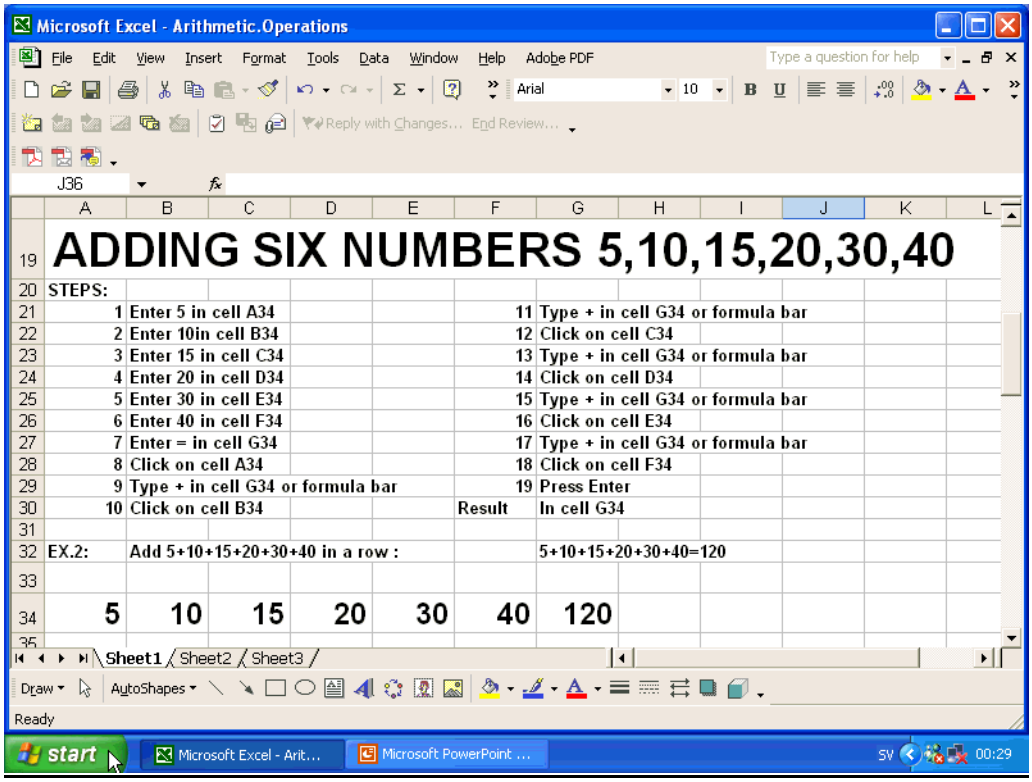
1. Open a blank worksheet.
2. Click on a cell where you would like to enter the number 10. Say cell A15.
3. Enter 10 in cell A15.
4. Click cell where you would like to enter the number 5. Say cell B15.
5. Click cell where you would like to get the sum of 10 and 5. Say cell C15.
6. Start the formula. Write equal sign = in cell C15.
7. After =, write "(" (left bracket) in cell C15.
8. Move mouse and left click on value 10 which is in cell A15. In cell C15, the cell reference A15 is written.
9. Write "+" after "A15" in cell C15.
10. Move mouse and left click on value 5 which is in cell B15. In cell C15, the cell reference B15 is written.
11. Write ") " (right bracket) in cell C15.
12. Press Enter key

The answer 15 is shown in cell C15.
If you click on cell C15, the formula "=A15+B15" is displayed the formula bar to the right of fx in the Toolbar.

The main steps along with the entries are shown in the slide below. The worksheet MTH302-lec-02 contains the actual entries.



The next slide shows addition of 6 numbers 5, 10, 15, 20, 30 and 40. The entries were made in row 34. The values were entered as follows:
Cell A34: 5
Cell B34: 10
Cell C34: 15
Cell D34: 20
Cell E34: 30
Cell F34: 40
The formula was written in cell G34. The formula was:
=5+10+15+20+30+40
The answer was 120.
You can use an Excel function SUM along with the cell range A34:F34 to calculate the sum of the above numbers. The formula in such a case will be:
=SUM(A34:F34)
You enter "=" followed by SUM, followed by "(". Click on the cell with value 5(reference: A34). Drag the mouse to cell with value 40(reference: F34) and drop the mouse. Enter ")" and then press the Enter key.



In the above two examples you learnt how formulas for addition are written in Excel.

Excel Formula for Subtraction

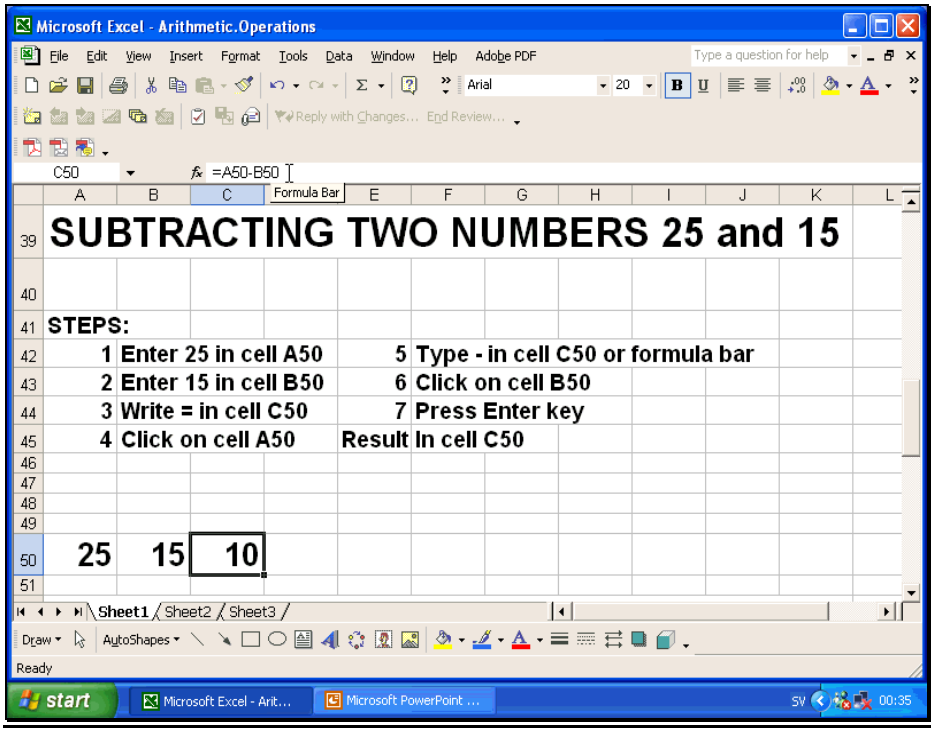
Excel formulas for subtraction are similar to those of addition but with the minus sign. Let us go through the steps for subtracting 15 from 25. Enter values in row 50 as follows:

Cell A50: 25
Cell B50: 15

Write the formula in cell C50 as follows:
=A50-B50

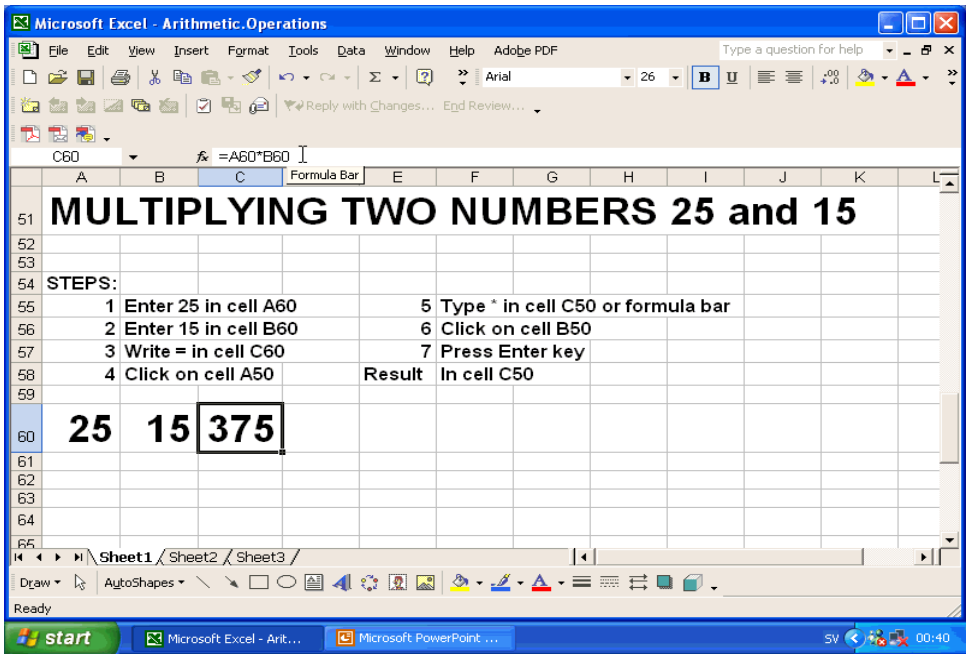
To write this formula, click cell C50, where you want the result. Enter "=". Click on cell with value 25 (reference:A50). Enter "-"(minus sign). Click on cell with value 15 (reference B50). Press enter key.

If you enter 15 first and 25 later, then the question will be to find result of subtraction 15-25.



Excel Formula for Multiplication

Excel formula for multiplication is also similar to the formula for addition. Only the sign of multiplication will be used. The Excel multiplication operator is *.



Let us look at the multiplication of two numbers 25 and 15. The entries will be made in row 60. Enter values as under:

Cell A50: 25

Cell B50: 15

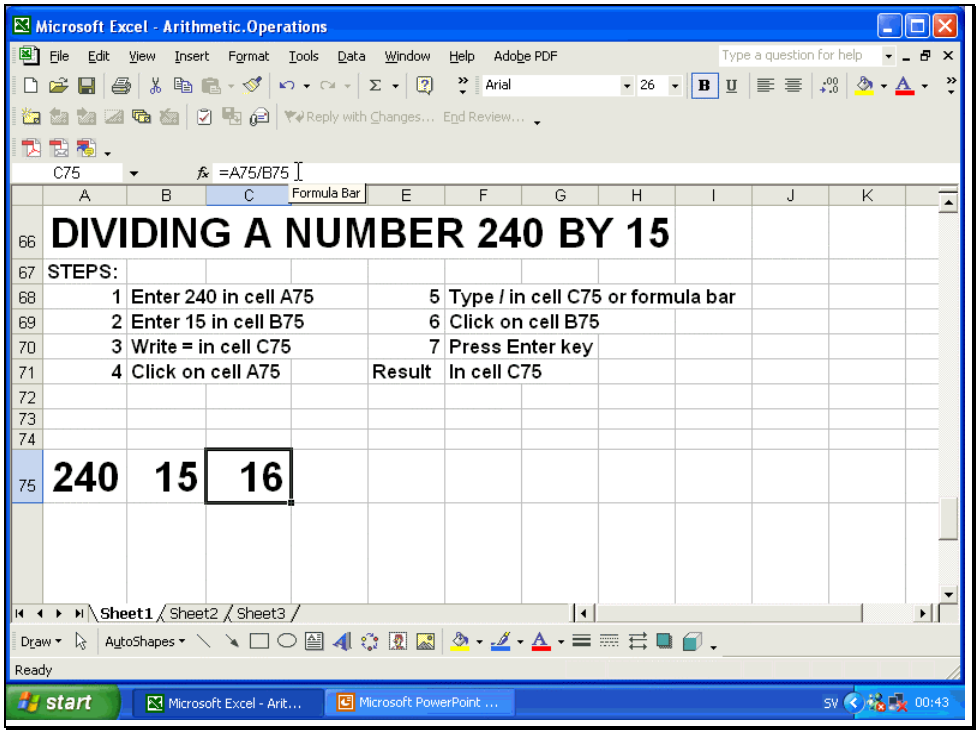
The formula for multiplication is:

=A50*B50

Click on cell C50 to write the formula in that cell. Enter "=". Click on cell with number 25 (reference: A50). Enter "*". Click on cell with number 15 (reference: B50). Press Enter key. The answer is 375 in cell C50.

Excel Formula for Division

The formula for division is similar to that of multiplication with the difference that the division sign "/" will be used.



Let us divide 240 by 15using Excel formula for division. Let us enter numbers in row 75 as follows:

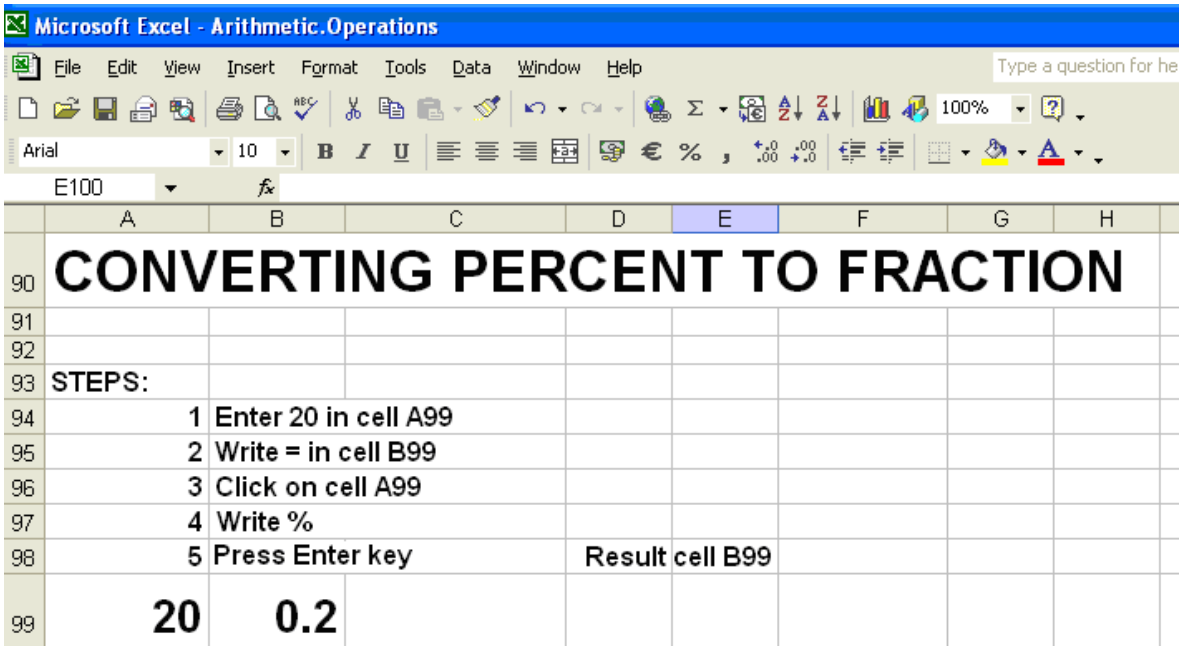
Cell A75: 240
Cell B75: 15

The formula for division will be written in cell C75 as under:
=A75/B75

The steps are as follows: Click the cell A75. Enter 240 in cell A75. Click cell B75. Enter 15. Click cell C75. Enter "=". Click on cell with value 240 (reference: A75). Enter "/". Click cell with number 15 (reference: B75). Press enter key. The answer 16 will be displayed in cell C75.

Excel Formula for Percent

The formula for converting percent to fraction uses the symbol %. To convert 20% to fraction the formula is as under:
=20%
If you enter 20 in cell A99, you can write formula for conversion to fraction by doing the following:
Enter 2o in cell A99. In cell B99 enter "=". Click on cell A99. Enter"%". Press Enter key. The answer 0.2 is given in cell B99.



Excel Formula for Exponentiation

The symbol for exponentiation is ^. The formula for calculating exponents is similar to multiplication with the difference that the carat symbol ^ will be used.

Let us calculate 16 raised to the power 2 by Excel formula for exponentiation. The values will be entered in row 85.

The steps are:

Select Cell A85. Enter 16 in this cell.

Select cell B85 Enter 2 in this cell.

Select cell C85.

Enter"=".

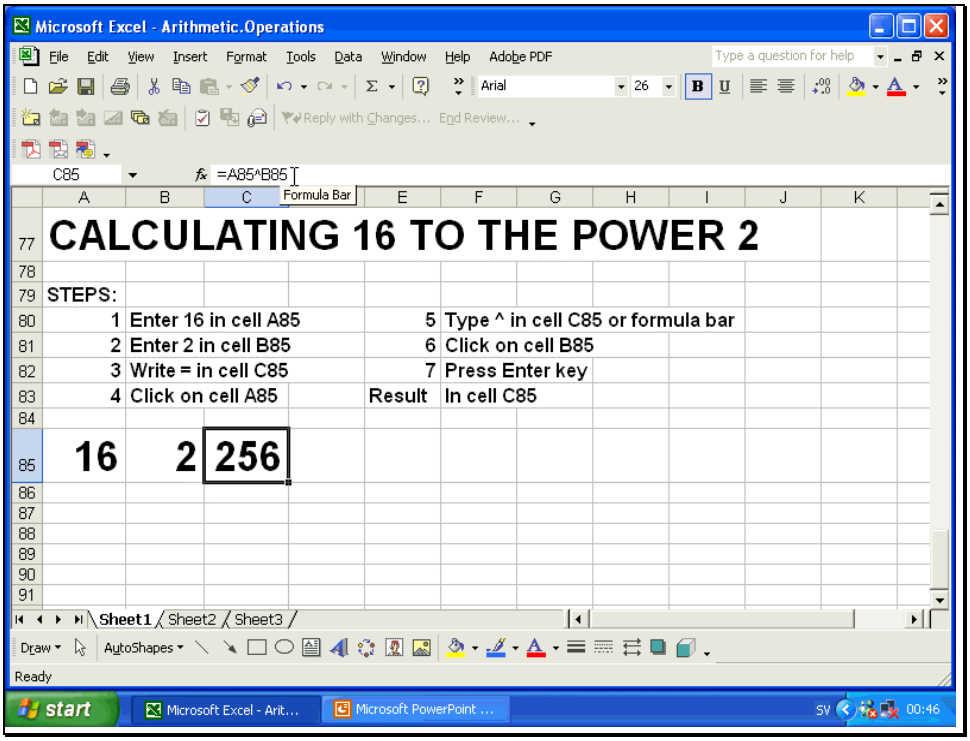
Select cell with value 16 (reference:A85).

Enter "^".

Select number 2 (reference: B85)

Press Enter key.

The result 256 is displayed in cell C85.



Recommended Homework

Download worksheet MTH302-lec-02.xls from the course web site.
Change values to see change in results.
Set up new worksheets for each Excel operator with different values.
Set up worksheets with combinations of operations.

LECTURE # 16

Applications of Basic Mathematics(Continued)

OBJECTIVES

The objectives of the lecture are to learn about:
Basic calculations of percentages, salaries and investments using
Microsoft Excel

PERCENTAGE CHANGE

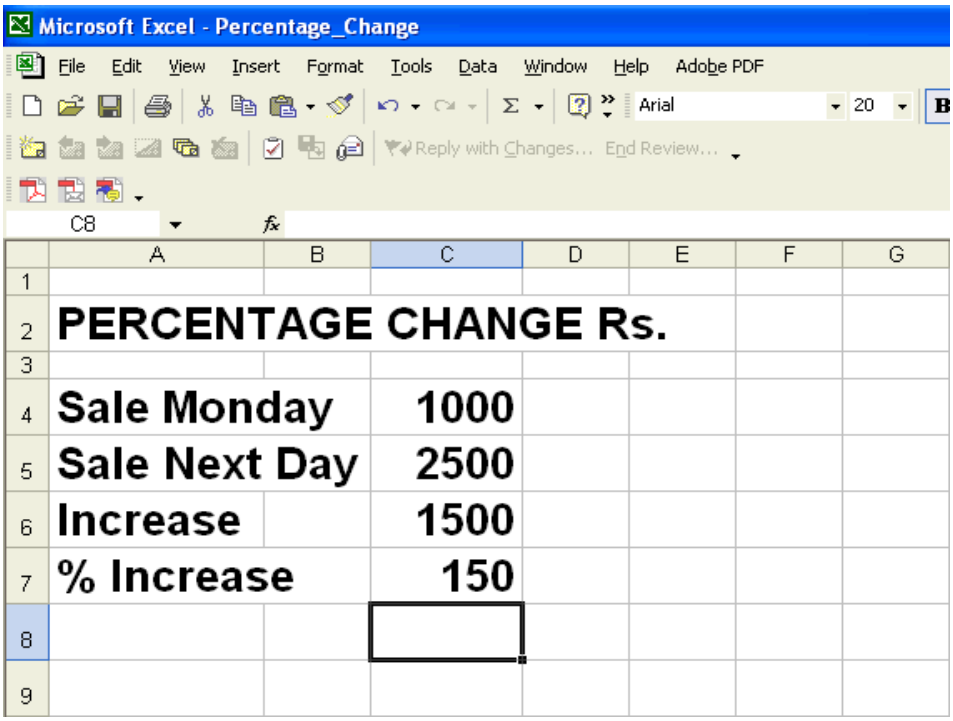
Monday's Sales were Rs.1000 and grew to Rs. 2500 the next day.
Find the percent change.

METHOD

Change = Final value – initial value
Percentage change = (Change / initial value) x 100%

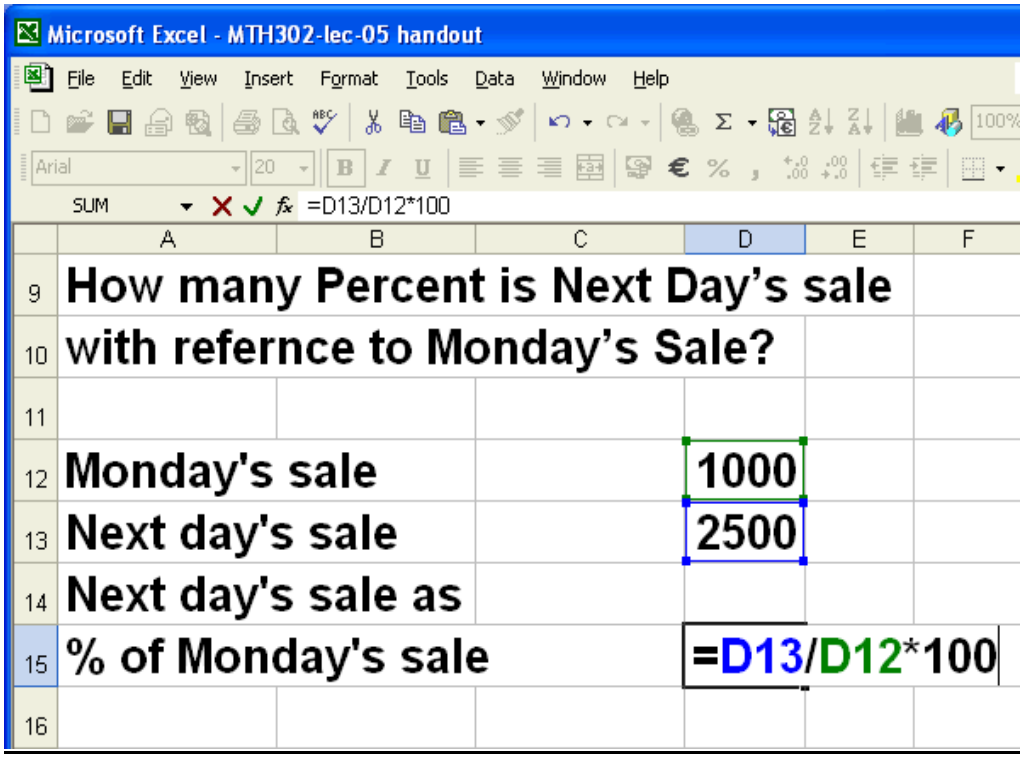
CALCULATION:-

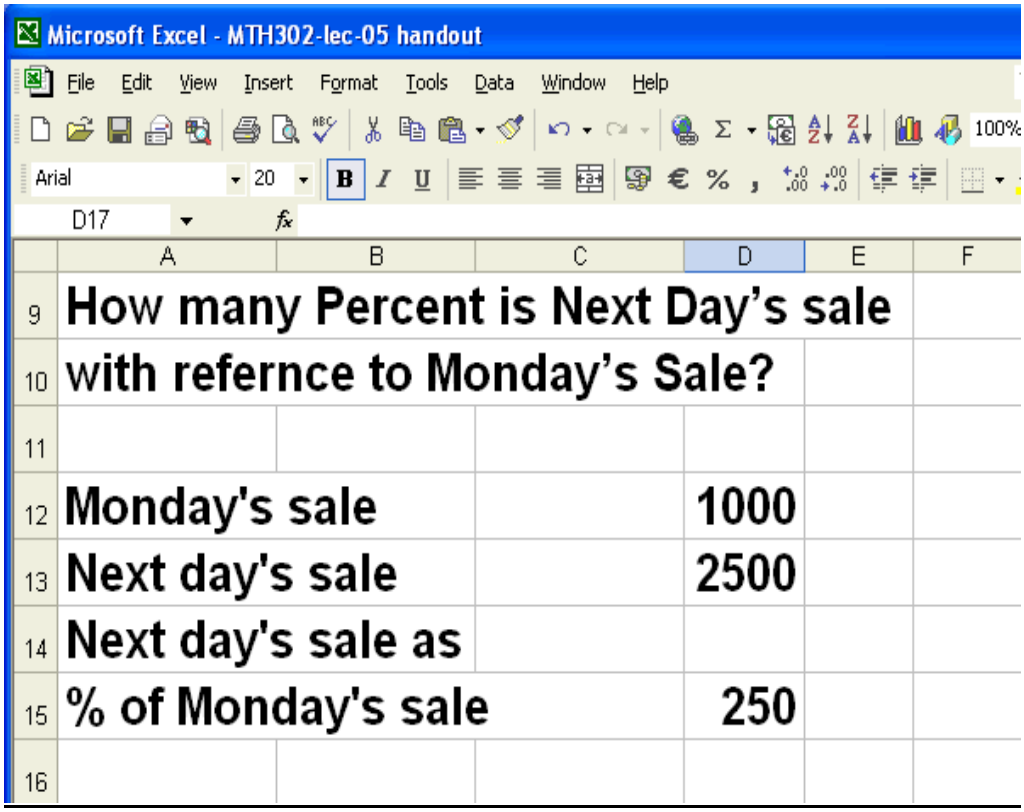
Initial value =1000
Final value = 2500
Change = 1500
% Change = (1500/1000) x 100 = 150%
The calculations using Excel are given below.
First the entries of data were made as follows:
Cell C4 = 1000
Cell C5 = 2500
In cell C6 the formula for increase was: = C5 – C4
The result was 1500.
In cell C7 the formula for percentage change was: = C6/C4*100
The result 150 is shown in the next slide.



EXAMPLE 1:-

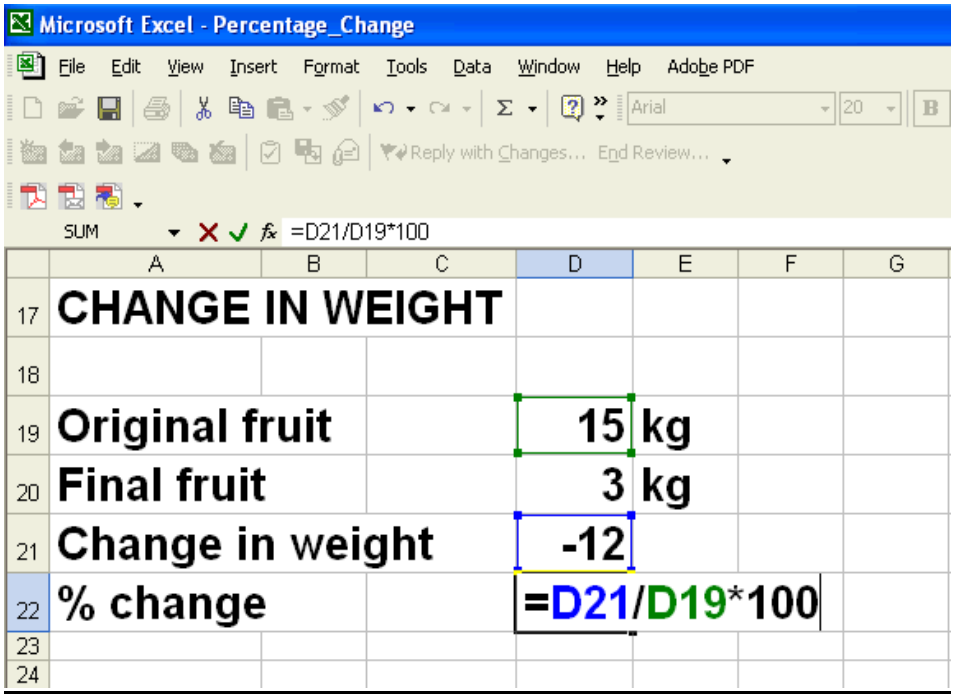
How many Percent is Next Day's sale with reference to Monday's Sale?
Monday's sale= 1000
Next day's sale= 2500
Next day's sale as % = 2500/1000 x 100 = 250 %
= Two and a half times

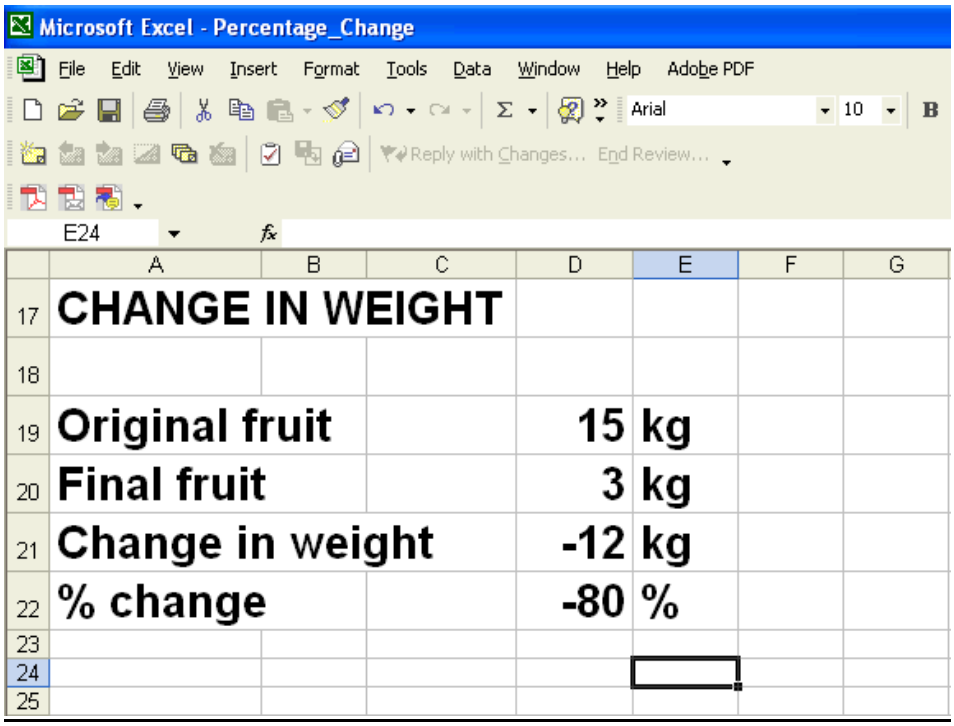




EXAMPLE 2:-
In the making of dried fruit, 15kg. of fresh fruit shrinks to 3 kg of dried fruit.
Find the percentage change.

Calculation
Original fruit = 15 kg
Final fruit = 3 kg
Change = 3-15 = -12
% change = - 12/15 x 100 = - 80 %
Size was reduced by 80%





Calculations in Excel were done as follows:

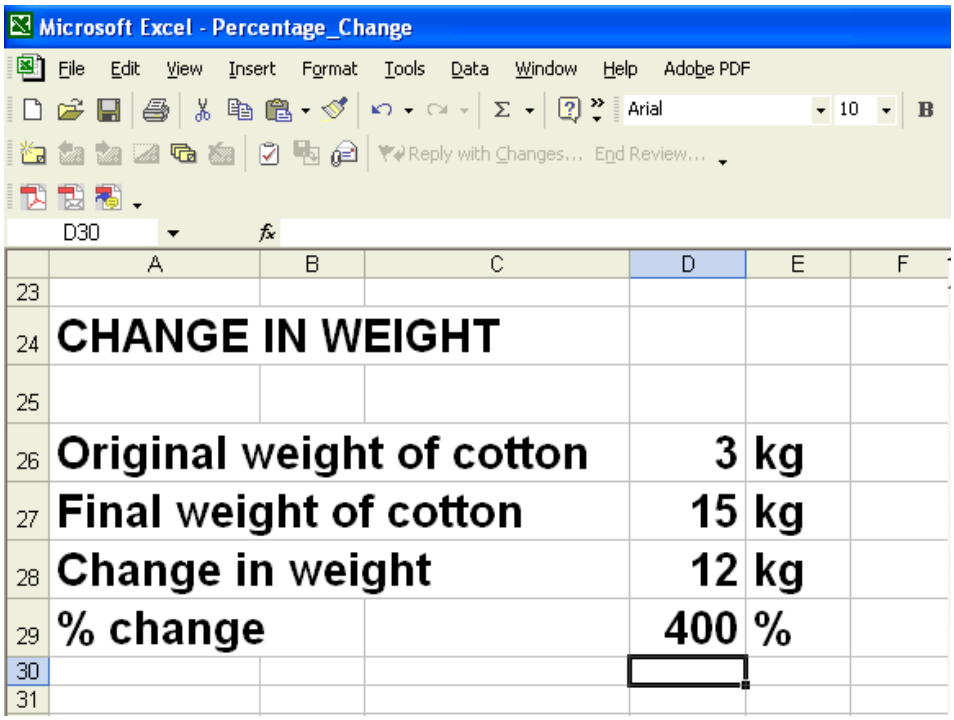
Data entry
Cell D19: 15
Cell D20: 3

Formulas
Formula for change in Cell D21: = D20 – D19
Formula for %change in Cell D22: = D21/D19*100

Results
Cell D21 = -12 kg
Cell D22 = -80 %

EXAMPLE 3
After mixing with water the weight of cotton increased from 3 kg to 15 kg. Find the percentage change.

CALCULATION
Original weight = 3 kg
Final weight = 15 kg
Change = 15-3= 12
% change = 12/3 x 100 = 400 %
Weight increased by 400%



Calculations in Excel were done as follows:

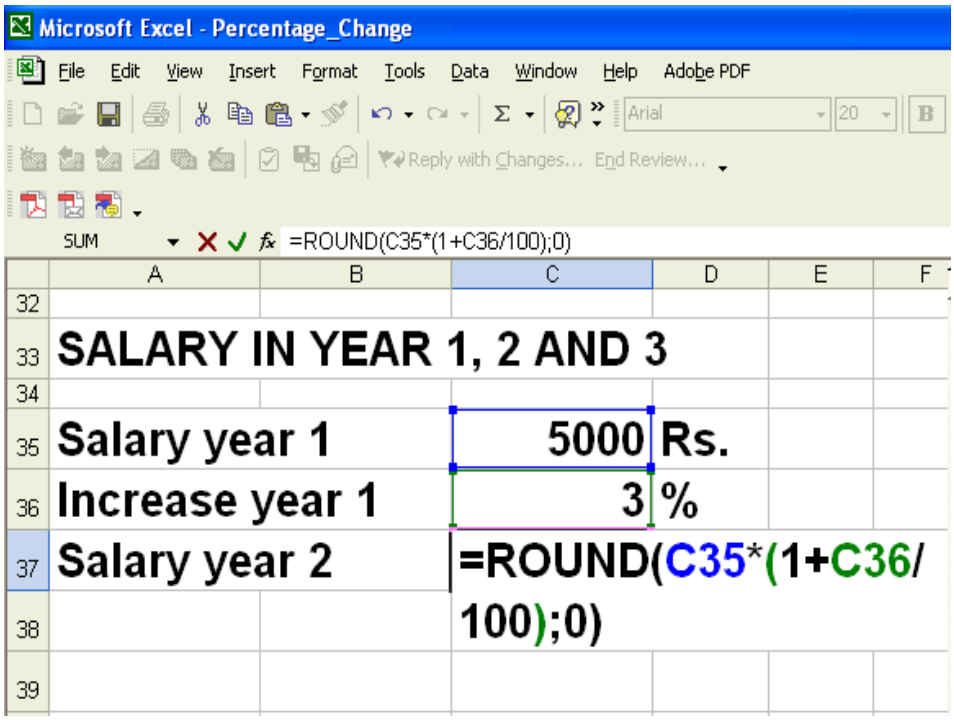
Data entry
Cell D26: 3
Cell D27: 15

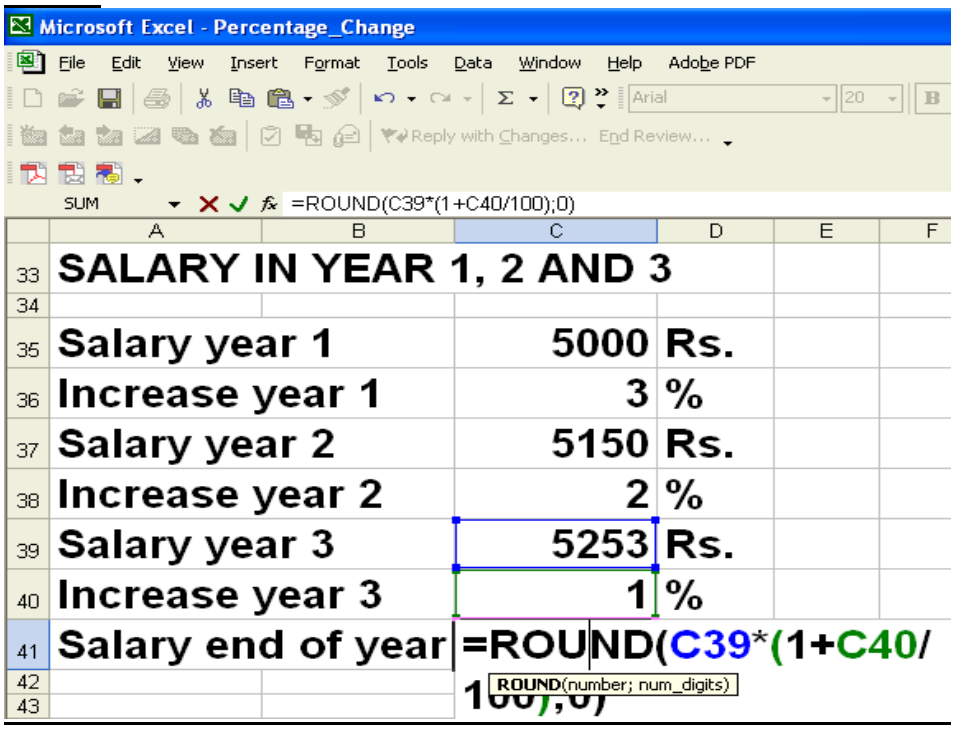
Formulas
Formula for change in Cell D28: = D27 – D26
Formula for %change in Cell D29: = D28/D26*100

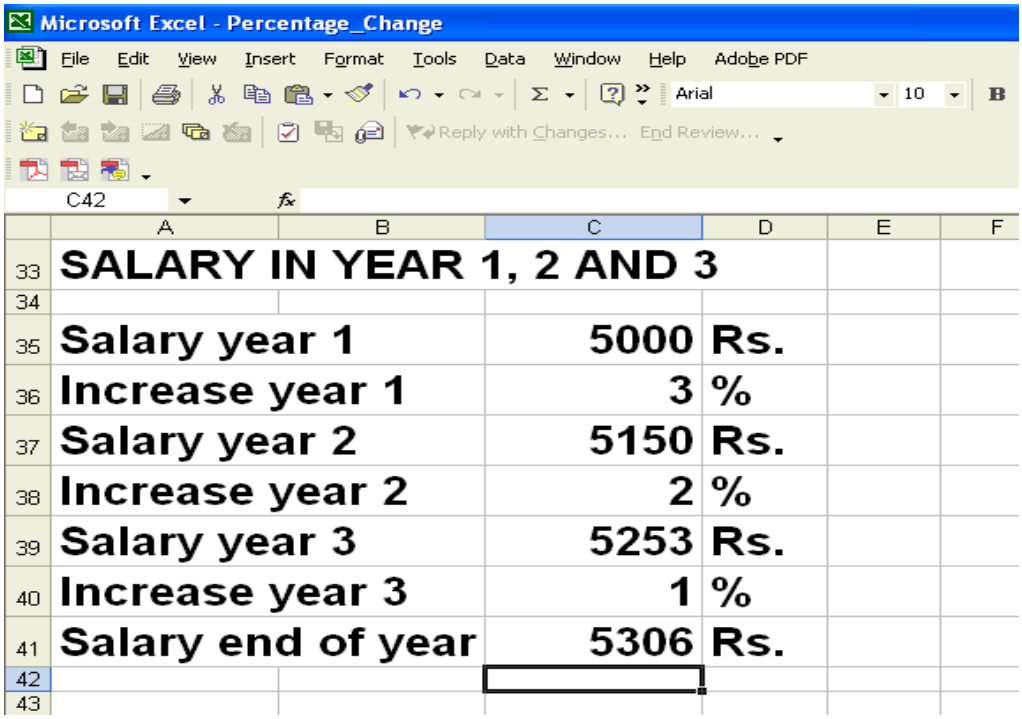
Results
Cell D28 = 12 kg
Cell D29 = 400 %

EXAMPLE 4:-
A union signed a three year collective agreement that provided for wage increases of 3%, 2%, and 1% in successive years
An employee is currently earning 5000 rupees per month
What will be the salary per month at the end of the term of the contract?

Calculation:-
= 5000(1 + 3%)(1 + 2%)(1 + 1%)
= 5000 x 1.03 x 1.02 x 1.01
= 5306 Rs.
Calculations using Excel are shown in the following slides.







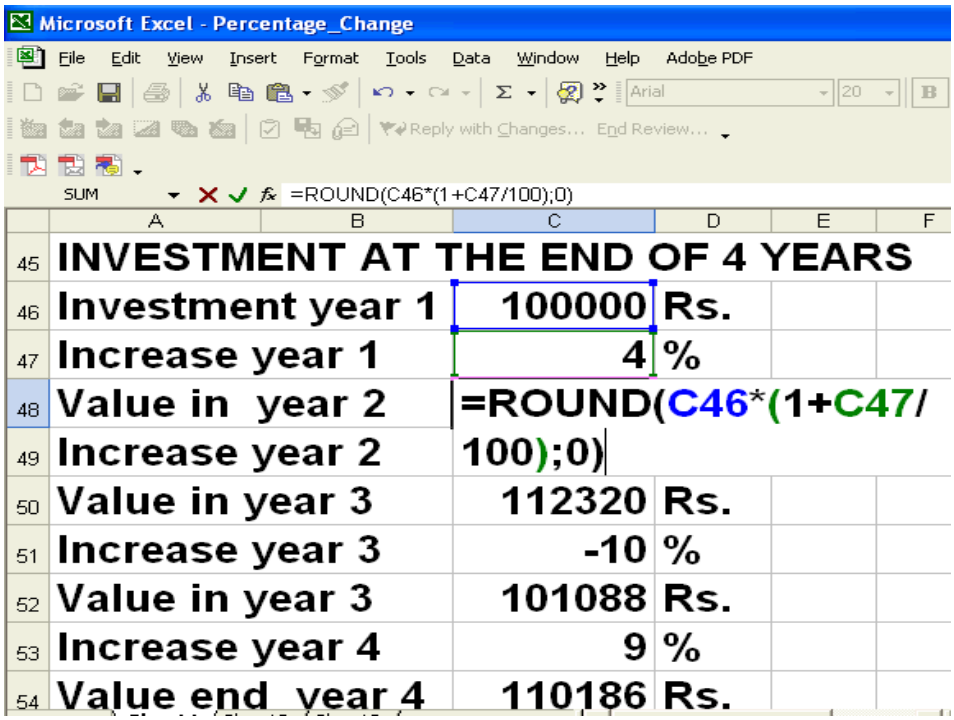
Calculations in Excel were done as follows:

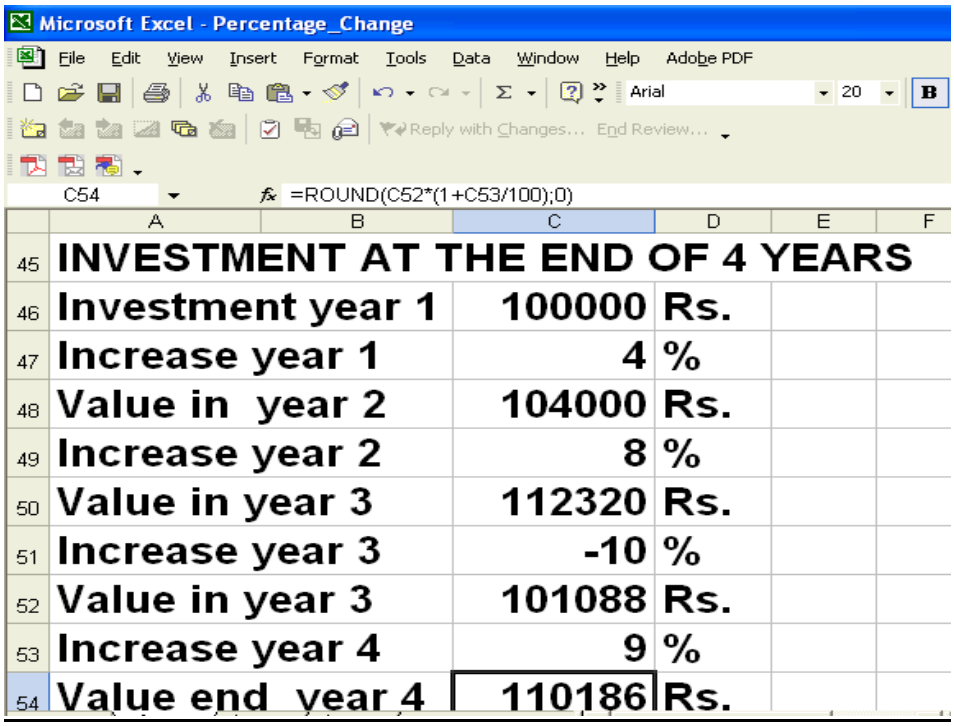
Data entry
Cell C35: 5000
Cell C36: 3
Cell C38: 2
Cell C40: 1

Formulas
Formula for salary in year 2 in Cell C37: =ROUND(C35*(1+C36/100);0)
Formula for salary year 3 in Cell C39: =ROUND(C37*(1+C38/100);0)
Formula for salary at the end of year 3 in Cell C41: =ROUND(C35 C39*(1+C39/100);0)

Results
Cell C37 = 5150 Rs.
Cell C39 = 5253 Rs.
Cell C41= 5306 Rs.

EXAMPLE 5
An investment has been made for a period of 4 years.
Rates of return for each year are 4%, 8%, -10% and 9% respectively. If you invested Rs. 100,000 at the beginning of the term, how much will you have at the end of the last year?





	A	B	C	D	E	F
45	INVESTMENT AT THE END OF 4 YEARS					
46	Investment year 1		100000	Rs.		
47	Increase year 1		4	%		
48	Value in year 2		104000	Rs.		
49	Increase year 2		8	%		
50	Value in year 3		112320	Rs.		
51	Increase year 3		-10	%		
52	Value in year 3		101088	Rs.		
53	Increase year 4		9	%		
54	Value end year 4		110186	Rs.		

Calculations in Excel were done as follows:

Data entry
Cell C46: 100000
Cell C47: 4
Cell C49: 8
Cell C51: -10
Cell C53: 9

Formulas
Formula for value in year 2 in Cell C48: = ROUND(C46*(1+C47/100);0)
Formula for value in year 3 in Cell C50: = ROUND(C48*(1+C49/100);0)
Formula for value in year 4 in Cell C52: = ROUND(C50*(1+C51/100);0)
Formula for salary end of year 4 in Cell C54: = ROUND(C52*(1+C53/100);0)

Results
Cell C48 = 104000 Rs.
Cell C50 = 112320 Rs.
Cell C52 = 101088 Rs.
Cell C54 = 110186Rs.

LECTURE # 17

Applications of Basic Mathematics(Continued)

OBJECTIVES

The objectives of the lecture are to learn about:

- Discount
- Simple and compound interest
- Average due date, interest on drawings and calendar

REVISION LECTURE

A chartered bank is lowering the interest rate on its loans from 9% to 7%.
What will be the percent decrease in the interest rate on a given balance?
A chartered bank is increasing the interest rate on its loans from 7% to 9%
What will be the percent increase in the interest rate on a given balance?
As we learnt in lecture 5, the calculation will be as follows:
Decrease in interest rate = 7-9 = -2 %
% decrease = -2/9 x 100 = -22.2 %
Increase in interest rate = 9-7 = 2 %
% decrease = 2/7 x 100 = 28.6 %
The calculations in Excel are shown in the following slides:

DECREASE IN RATE
Data entry
Cell F4 = 9
Cell F5 = 7

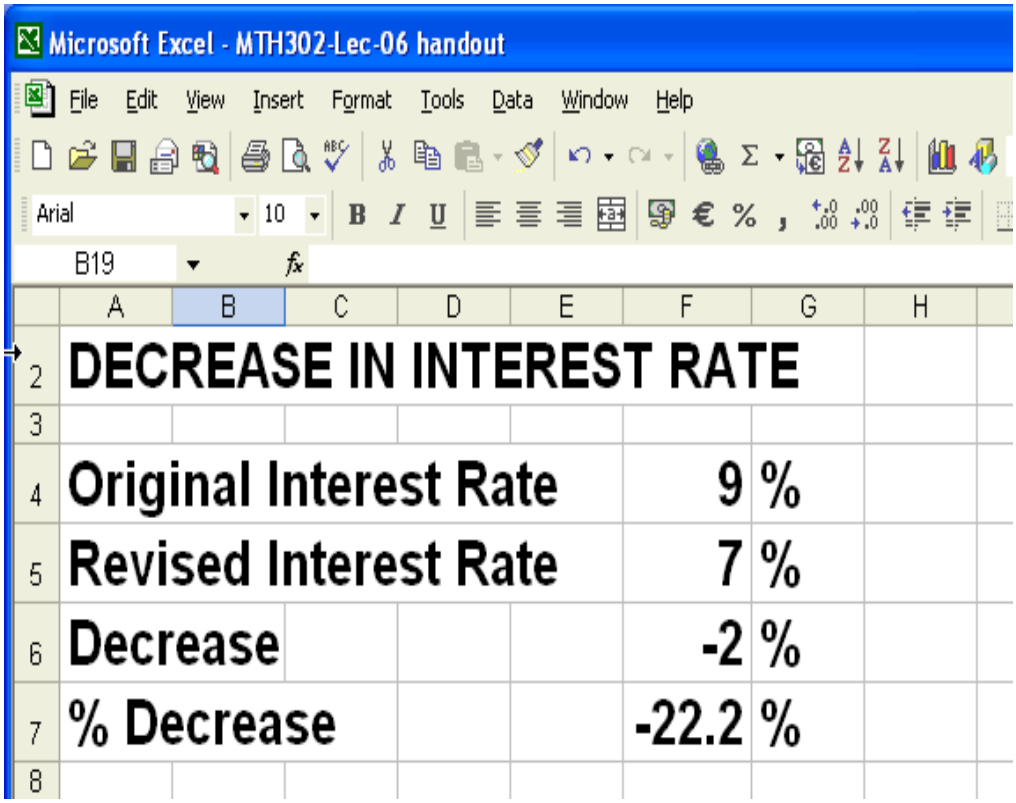
Formulas
Formula for decrease in Cell F6: = =F5-F4
Formula for % decrease in Cell F7: =F6/F4*100

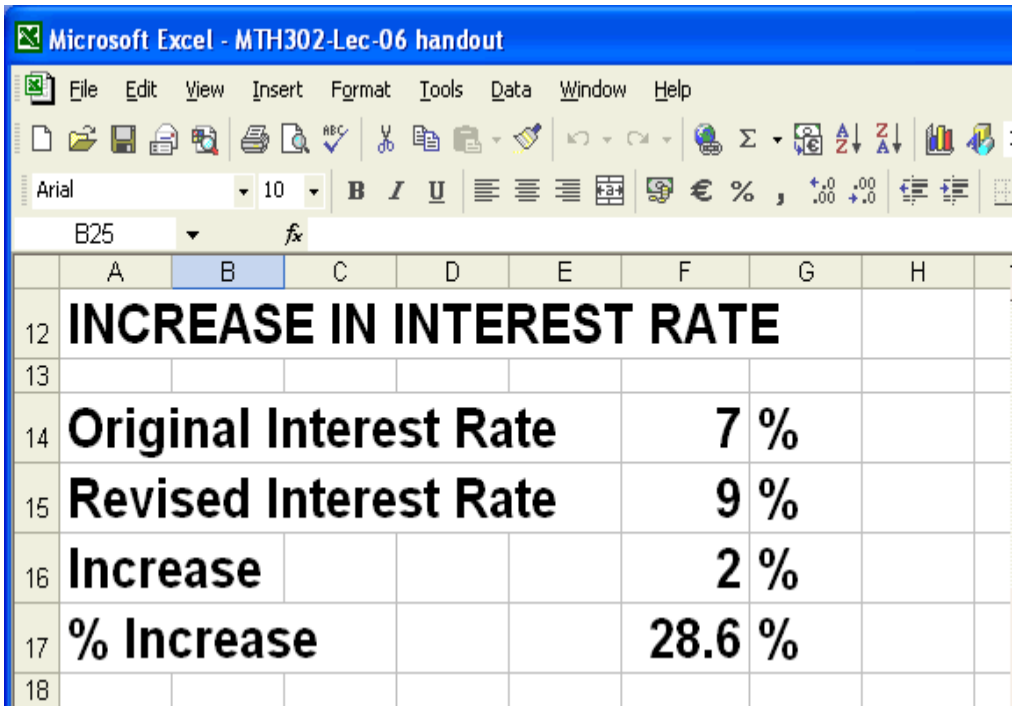
Results
Cell F6 = -2%
Cell F7 = -22.2%

INCREASE IN RATE
Data entry
Cell F14 = 7
Cell F15 = 9

Formulas
Formula for increase in Cell F16: =F15-F14
Formula for % increase in Cell F17: =F16/F14*100

Results
Cell F16 = 2%
Cell F17= 28.6%





The Definition of a Stock

Plain and simple, stock is a share in the ownership of a company. Stock represents a claim on the company's assets and earnings. As you acquire more stock, your ownership

stake in the company becomes greater. Whether you say shares, equity, or stock, it all means the same thing.

Stock yield

With stocks, yield can refer to the rate of income generated from a stock in the form of regular dividends. This is often represented in percentage form, calculated as the annual dividend payments divided by the stock's current share price.

Earnings per share (EPS)

The EPS is the total profits of a company divided by the number of shares. A company with $1 billion in earnings and 200 million shares would have earnings of $5 per share.

Price-earnings ratio

A valuation ratio of a company's current share price compared to its per-share earnings.

Calculated as:

$$= \frac{\text{Market Value per Share}}{\text{Earnings per Share (EPS)}}$$

For example, if a company is currently trading at $43 a share and earnings over the last 12 months were $1.95 per share, the P/E ratio for the stock would be 22.05 ($43/$1.95).

Outstanding shares

Stock currently held by investors, including restricted shares owned by the company's officers and insiders, as well as those held by the public. Shares that have been repurchased by the company are not considered outstanding stock.

Net current asset value per share(NCAVPS)

NCAVPS is calculated by taking a company's current assets and subtracting the total liabilities, and then dividing the result by the total number of shares outstanding.

$$\text{NCAVPS} = \frac{(\text{Current Assets} - \text{Total Liabilities})}{\#\,\text{Of Shares Outstanding}}$$

Current Assets

The value of all assets that are reasonably expected to be converted into cash within one year in the normal course of business. Current assets include cash, accounts receivable, inventory, marketable securities, prepaid expenses and other liquid assets that can be readily converted to cash.

Liabilities

A company's legal debts or obligations that arise during the course of business operations.

Market value

The price at which investors buy or sell a share of stock at a given time

Face value

Original cost of a share of stock which is shown on the certificate. Also referred to as "par value." Face value is usually a very small amount that bears no relationship to its market price.

Dividend

Usually, a company distributes a part of the profit it earns as dividend.

For example: A company may have earned a profit of Rs 1 crore in 2003-04. It keeps half that amount within the company. This will be utilised on buying new machinery or more raw materials or even to reduce its borrowing from the bank. It distributes the other half as dividend.

Assume that the capital of this company is divided into 10,000 shares. That would mean half the profit -- ie Rs 50 lakh (Rs 5 million) -- would be divided by 10,000 shares; each share would earn Rs 500. The dividend would then be Rs 500 per share. If you own 100

shares of the company, you will get a cheque of Rs 50,000 (100 shares x Rs 500) from the company.

Sometimes, the dividend is given as a percentage -- i e the company says it has declared a dividend of 50 percent. It's important to remember that this dividend is a percentage of the share's face value. This means, if the face value of your share is Rs 10, a 50 percent dividend will mean a dividend of Rs 5 per share

BUYING SHARES

If you buy 100 shares at Rs. 62.50 per share with a 2% commission, calculate your total cost.

Calculation

100 * Rs. 62.50 = Rs. 6,250
0.02 * Rs. 6,250 = 125
Total = Rs. 6,375

RETURN ON INVESTMENT

Suppose you bought 100 shares at Rs. 52.25 and sold them after 1 year at Rs. 68. With a 1% commission rate of buying and selling the stock and 10 % dividend per share is due on these shares. Face value of each share is 10Rs. What is your return on investment?

Bought

100 shares at Rs. 52.25 = 5,225.00
Commission at 1% = 52.25
Total Costs =5,225 + 52.25 = 5,277.25
Sold
100 shares at Rs. 68 = 6,800.00
Commission at 1% = - 68.00
Total Costs Sale = 6,800 – 68 = 6,732.00
Gain
Net receipts = 6,732.00
Total cost = - 5,277.25
Net Gain = 6,732 – 5,277.25 =1,454.75
Dividends (100*10/10) = 100.00
Total Gain = 1,454.75 + 100 = 1,554.75
Return on investment = 1,554.75/5,277.25*100
= 29.46 %

The calculations using Excel were made as follows:
BOUGHT
Data entry

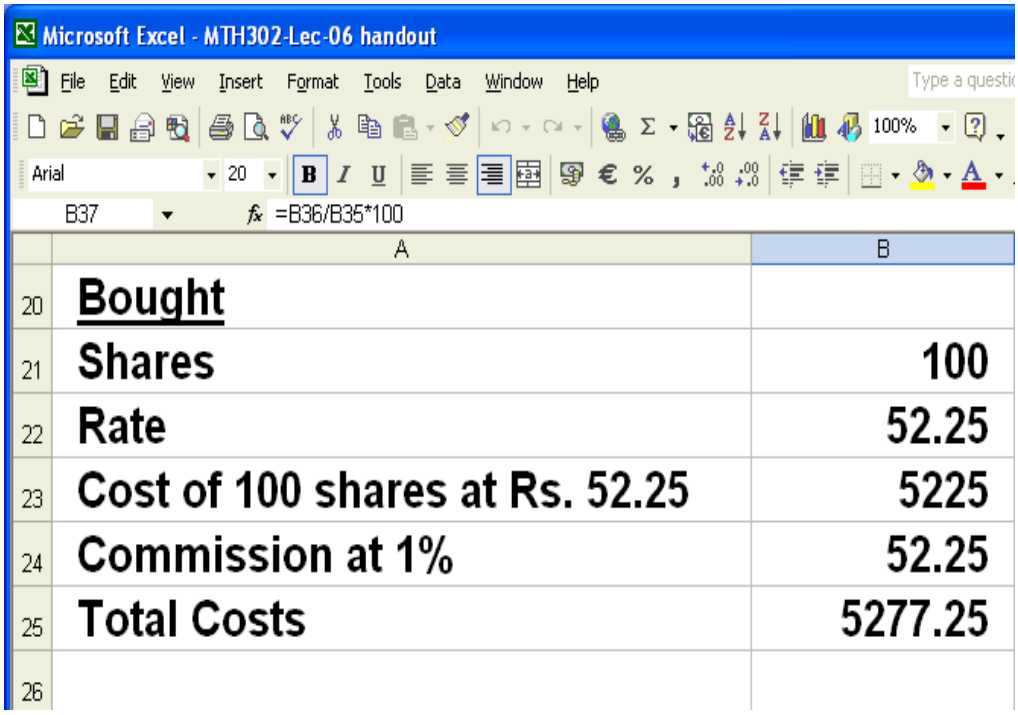
Cell B21: 100
Cell B22: 52.25

Formulas

Formula for Cost of 100 shares at Rs. 52.25 in Cell B23: =B21*B22
Formula for Commission at 1% in Cell B24: =B23*0.01
Formula for Total Costs in Cell B25: =B23+B24

Results

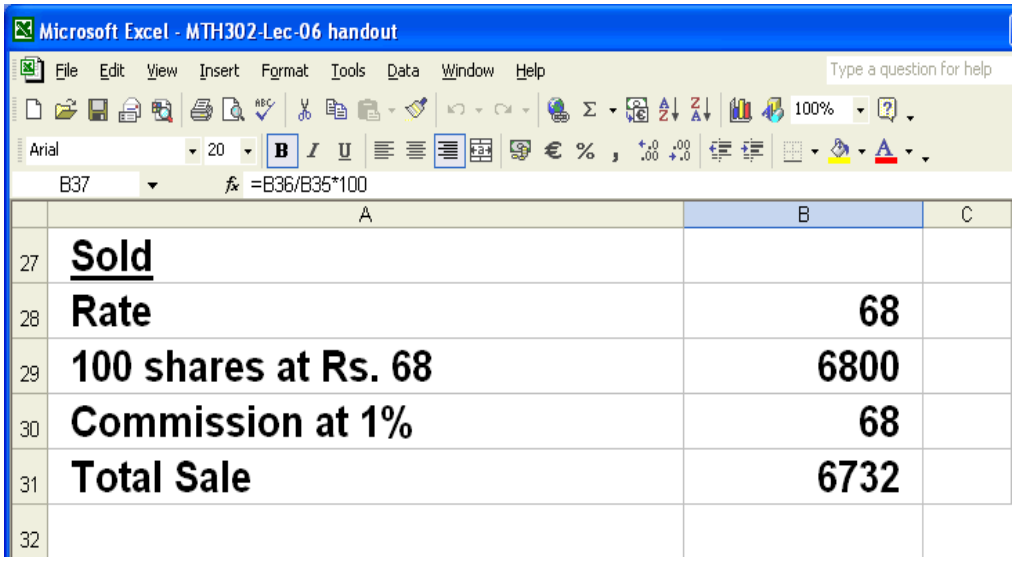
Cell B23 = 5225
Cell B24 = 52.25
Cell B25 = 5277.25



SOLD
Data entry
Cell B28: 68

Formulas
Formula for sale of 100 shares at Rs. 68 in Cell B29: =B21*B28
Formula for Commission at 1% in Cell B30: =B29*0.01
Formula for Total Sale in Cell B31: =B29-B30

Results
Cell B29 = 6800
Cell B30 = 68
Cell B31 = 6732



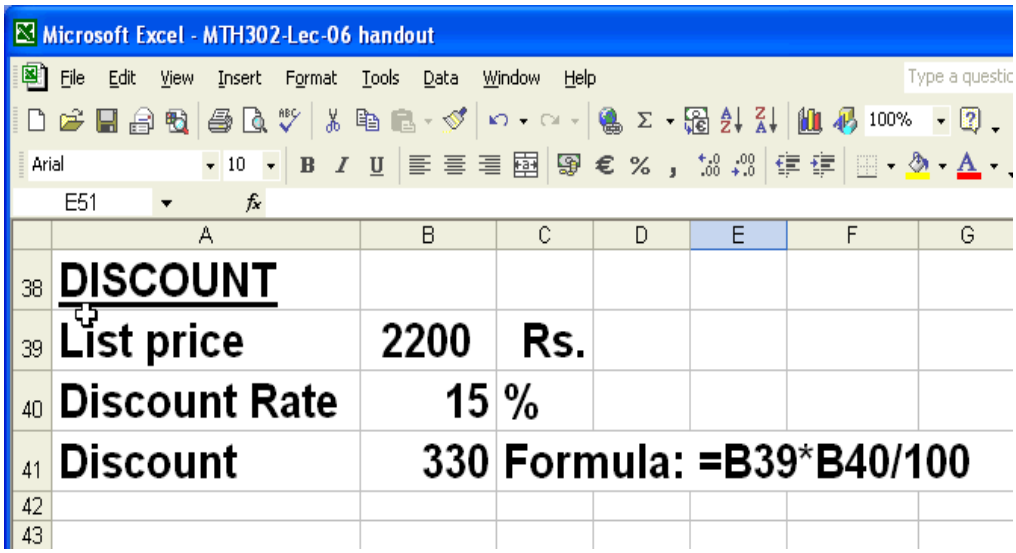
GAIN

Formulas
Formula for Net receipts in Cell B34: =B31
Formula for Total cost in Cell B35: =B25
Formula for Net Gain in Cell B36: =B31-B25
Formula for % Gain in Cell B37: =B36/B35*100

Results
Cell B34 = 6732
Cell B35 = 5277.25
Cell B36 = 1454.75
Cell B37 = 27.57

DISCOUNT
Discount is Rebate or reduction in price.
Discount is expressed as % of list price.

Example
List price = 2200
Discount Rate = 15%
Discount?
= 2200 x 0.15= 330
Calculation using Excel along with formula is given in the following slide:

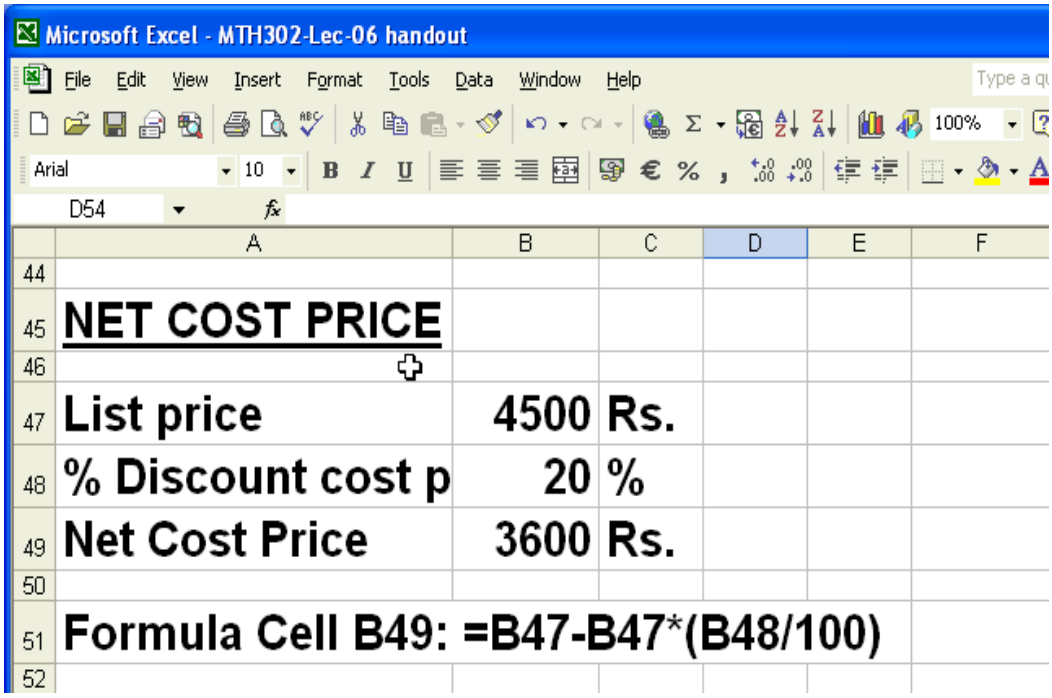


NET COST PRICE
Net Cost Price = List price - Discount

Example
List price = 4,500 Rs.
Discount = 20 %
Net cost price?

Net cost price = 4,500 – 20 % of 4,500
= 4,500 – 0.2 x4,500
=4,500 – 900
= 3,600 Rs.

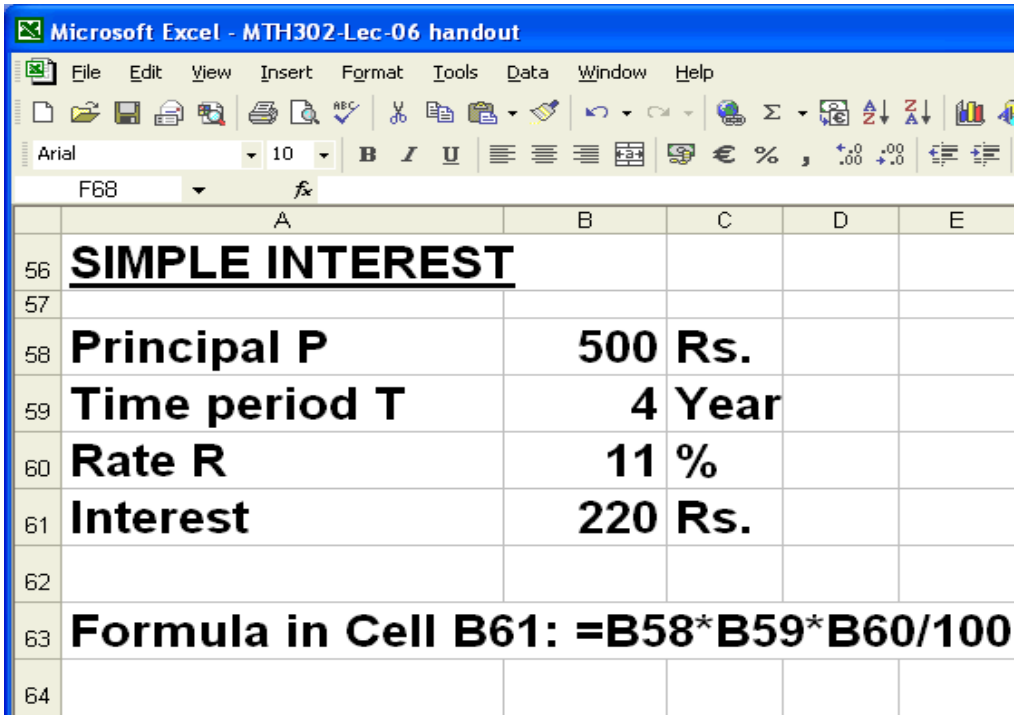
Calculation using Excel along with formula is given in the following slide:



SIMPLE INTEREST

P = Principal
R = Rate of interest per annum
T = Time in years
I = Simple interest
then
I = P. R. T / 100
Thus total amount A to be paid at the end of T years = P + I

Example
P = Rs. 500
T = 4 years
R =11%
Find simple interest
I = P x T x R /100
= 500 x 4 x 11/100
= Rs. 220
Calculation using Excel along with formula is given in the following slide:

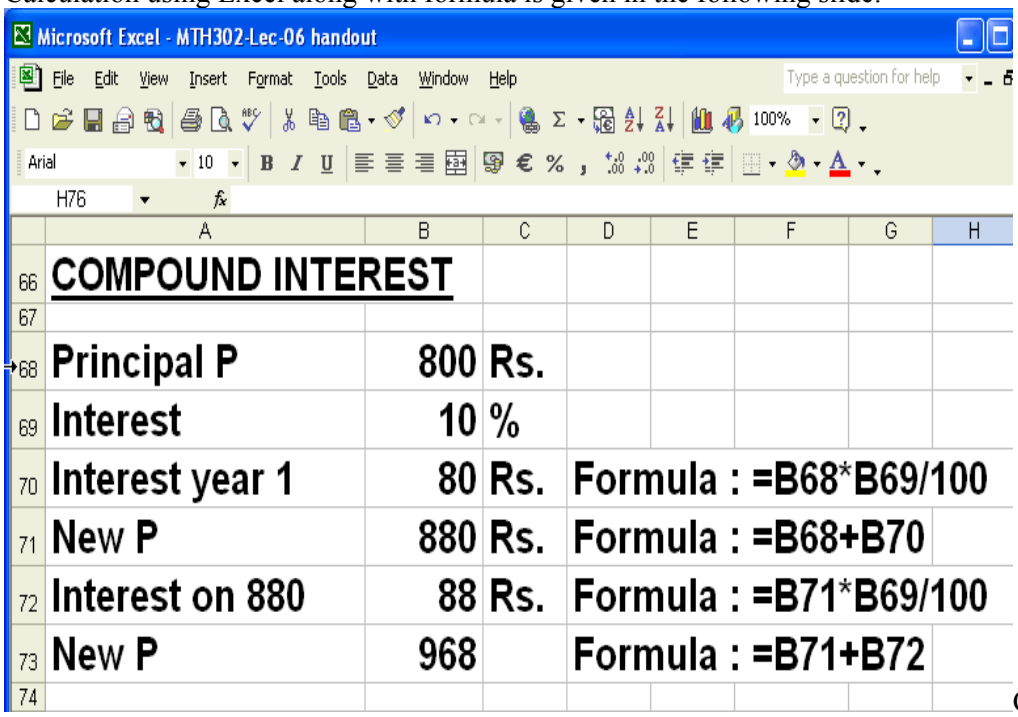


COMPOUND INTEREST

Compound Interest also attracts interest.

EXAMPLE:-

P = 800
Interest year 1= 0.1 x 800= 80
New P = 800 + 80 = 880
Interest on 880 = 0.1 X 880 = 88
New P = 880 + 88 = 968
Calculation using Excel along with formula is given in the following slide:

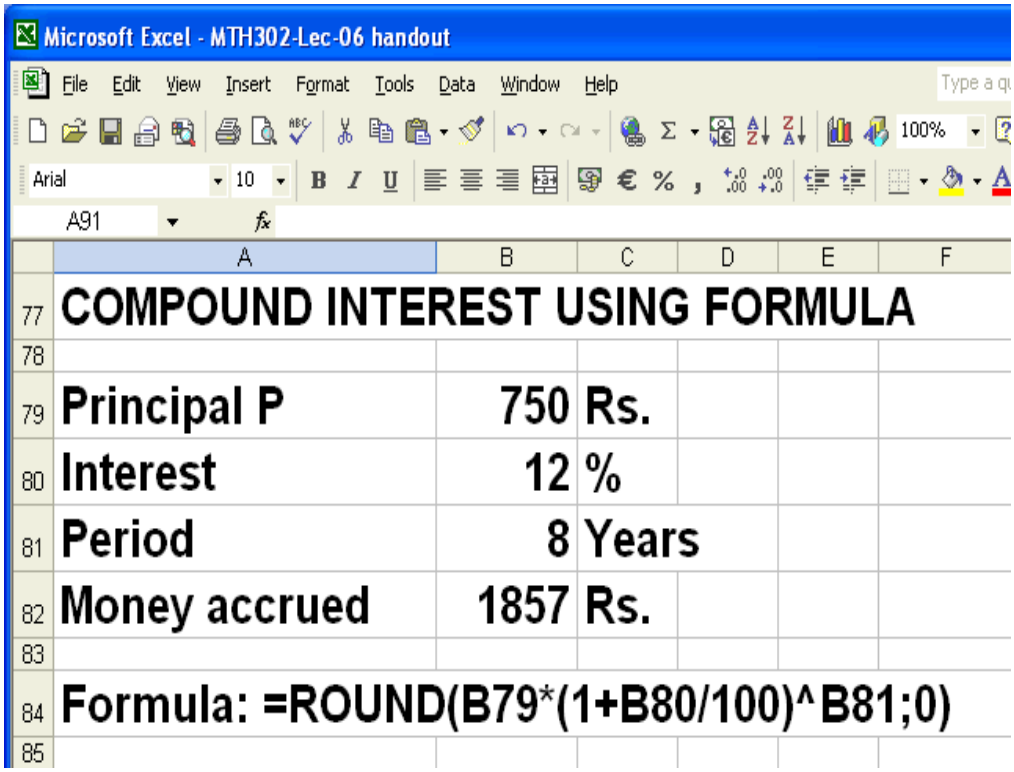


Compound Interest Formula
S = Money accrued after n years also called compound amount
P = Principal
r = Rate of interest
n = Number of periods
S = P(1 + r/100)^ n
Compound interest = S - P

Example

Calculate compound interest earned on Rs. 750 invested at 12% per annum for 8 years.
S= P(1+r/100)^8
= 750(1+12/100)^8
=1857 Rs
Compound interest = 1857 – 750 = 1107 Rs

Calculation using Excel along with formula is given in the following slide



LECTURE # 18

Applications of Basic Mathematics(Continued)

OBJECTIVES

The objectives of the lecture are to learn about:

- **Annuity**
- **Accumulated value**
- **Accumulation Factor**
- **Discount Factor**
- **Discounted value**
- **Algebraic operations**
- **Investments**
- **Exponents**
- **Solving Linear equations**

Annuity

It some point in your life you may have had to make a series of fixed payments over a period of time - such as rent or car payments - or have received a series of payments over a period of time, such as bond coupons. These are called annuities.

Annuities are essentially series of fixed payments required from you or paid to you at a specified frequency over the course of a fixed period of time.

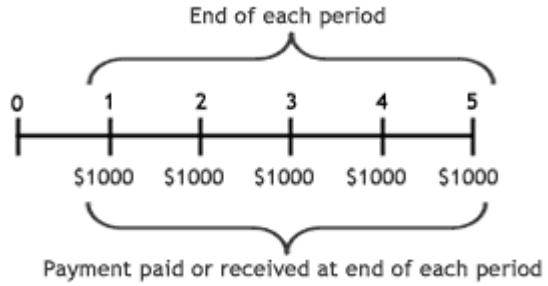
An annuity is a type of investment that can provide a steady stream of income over a long period of time. For this reason, annuities are typically used to build retirement income, although they can also be a tool to save for a child's education, create a trust fund, or provide for a surviving spouse or heirs.

The most common payment frequencies are yearly (once a year), semi-annually (twice a year), quarterly (four times a year) and monthly (once a month).

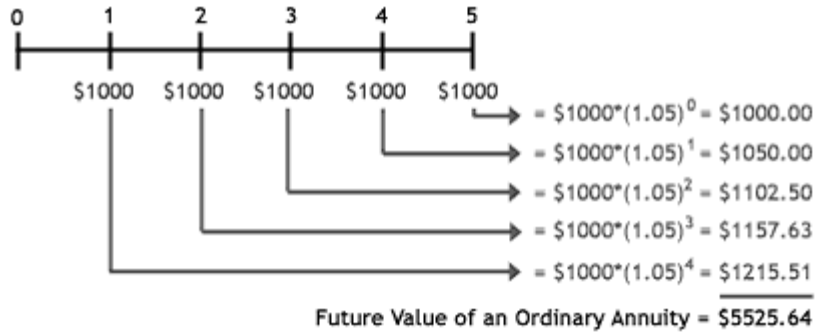
Calculating the Future Value or accumulated value of an Annuity

If you know how much you can invest per period for a certain time period, the future value of an ordinary annuity formula is useful for finding out how much you would have in the future by investing at your given interest rate. If you are making payments on a loan, the future value is useful for determining the total cost of the loan.

Let's now run through Example 1. Consider the following annuity cash flow schedule:



In order to calculate the future value of the annuity, we have to calculate the future value of each cash flow. Let's assume that you are receiving $1,000 every year for the next five years, and you invested each payment at 5%. The following diagram shows how much you would have at the end of the five-year period:



Since we have to add the future value of each payment, you may have noticed that, if you have an annuity with many cash flows, it would take a long time to calculate all the future values and then add them together. Fortunately, mathematics provides a formula that serves as a short cut for finding the accumulated value of all cash flows received from an annuity:

$$FV_{\text{Ordinary Annuity}} = C * \left[\frac{(1+i)^n - 1}{i}\right]$$

C=Payment per period or amount of annuity
i = interest rate
n = number of payments

((1 + i)n - 1) / i) is called accumulation factor for n periods.

Accumulated value of n period = payment per period × accumulation factor for n periods

If we were to use the above formula for Example 1 above, this is the result:

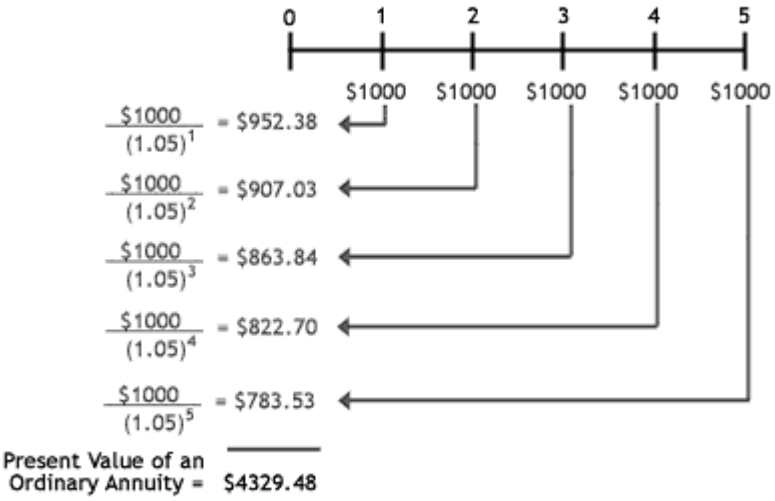
$$FV_{\text{Ordinary Annuity}} = \$1000 * \left[\frac{(1+0.05)^5 - 1}{0.05}\right]$$

=$1000*[5.53]
=$5525.63

Note that the $0.01 difference between $5,525.64 and $5,525.63 is due to a rounding error in the first calculation. Each of the values of the first calculation must be rounded to the nearest penny - the more you have to round numbers in a calculation the more likely rounding errors will occur. So, the above formula not only provides a short-cut to finding FV of an ordinary annuity but also gives a more accurate result.

Calculating the Present Value or discounted value of an Annuity
If you would like to determine today's value of a series of future payments, you need to use the formula that calculates the present value of an ordinary annuity.

For Example 2, we'll use the same annuity cash flow schedule as we did in Example 1. To obtain the total discounted value, we need to take the present value of each future payment and, as we did in Example 1, add the cash flows together.



Again, calculating and adding all these values will take a considerable amount of time, especially if we expect many future payments. As such, there is a mathematical shortcut we can use for PV of ordinary annuity.

$$PV_{\text{Ordinary Annuity}} = C * \left[\frac{1-(1+i)^{-n}}{i}\right]$$

C = Cash flow per period
i = interest rate
n = number of payments.
$(1 – (1 + i)^{-n}) / i$ is called discount factor for n periods.
Thus Discounted value of n period = payment per period × discount factor for n period
The formula provides us with the PV in a few easy steps. Here is the calculation of the annuity represented in the diagram for Example 2:

$$PV_{\text{Ordinary Annuity}} = \$1000 * \left[\frac{1-(1+0.05)^{-5}}{0.05}\right]$$

= $1000*[4.33]
= $4329.48

NOTATIONS
The following notations are used in calculations of Annuity:
R = Amount of annuity
N = Number of payments
I = Interest rater per conversion period
S = Accumulated value
A = Discounted or present worth of an annuity

ACCUMULATED VALUE
The accumulated value S of an annuity is the total payments made including the interest. The formula for Accumulated Value S is as follows:

S = r ((1+i)^n – 1)/i

Accumulation factor for n payments = ((1 + i)^n – 1) / i
It may be seen that:
Accumulated value = Payment per period x Accumulation factor for n payments

The discounted or present worth of an annuity is the value in today's rupee value.
As an example if we deposit 100 rupees and get 110 rupees

(i.e. 10 % interest on 100 Rs. which is 100*10/100 = 10 Rs. so total amount is 100+10 = 110 or
simply 100(1+10/100) = 100(1+ 0.1) = 100(1.1) = 110)

after one year, the Present Worth or of 110 rupees will be 100. Here 110 will be future value of 100 at the end of year 1.

The amount 110, if invested again, can be Rs. 121 after year 2.

(i.e. 10 % of 110 is 110*10/100 = 11, so total amount is 110+11 = 121)

The present value of Rs. 121, at the end of year 2, will also be 100.

DISCOUNT FACTOR AND DISCOUNTED VALUE

When future value is converted into present worth, the rate at which the calculations are made is called Discount factor rate. In the previous example 10% was used to make the calculations. This rate is called Discount Rate. The present worth of future payments is called Discounted Value.

EXAMPLE 1. ACCUMULATION FACTOR (AF) FOR n PAYMENTS

Calculate Accumulation Factor and Accumulated value when:
rate of interest i = 4.25 %
Number of periods n = 18
Amount of Annuity R = 10,000 Rs.
Accumulation Factor AF = ((1 + 0.0425)^18-1)/0.0425 = 26.24
Accumulated Value S = 10,000x 26.24 = 260,240 Rs

EXAMPLE 2. DISCOUNTED VALUE (DV)

In the above example calculate the value of all payments at the beginning of term of annuity i-e present value or discounted value.
Discount rate = 4.25%
Number of periods = 18
Amount of annuity= 10000 Rs
Value of all payments at the beginning of term of Annuity or discounted value
= Payment per period x Discount Factor (DF)
Formula for Discount Factor = (1-1/(1+i)^n)/i
= (1-1/(1+0.0425)^18))/0.0425= 12.4059
Discounted value = 10000 × 12.4059 = 124059 Rs

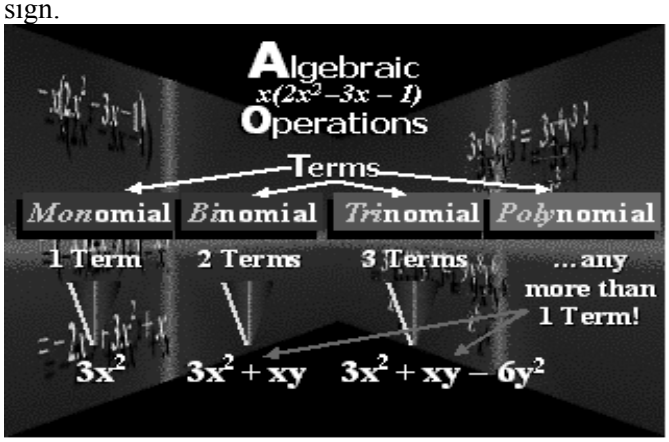
EXAMPLE 3. DISCOUNTED VALUE (DV)

How much money deposited now will provide payments of Rs. 2000 at the end of each half-year for 10 years if interest is 11% compounded six-monthly.
Amount of annuity = 2000Rs
Rate of interest = i = 11% / 2 = 0.055
Number of periods = n = 10 × 2 = 20
DISCOUNTED VALUE = 2,000 x ((1-1 / (1+0.055)^20) / 0.055)
= 2,000 x11.95
=23,900.77

ALGEBRAIC OPERATIONS

Algebraic Expression indicates the mathematical operations to be carried out on a combination of NUMBERS and VARIABLES.

The components of an algebraic expression are separated by Addition and Subtraction. In the expression 2x2 – 3x -1 the components 2x2, 3x and 1 are separated by minus "-" sign.



In algebraic expressions there are four types of terms:

- Monomial, i.e. 1 term (Example: 3x2)
- Binomial, i.e. 2 terms (Example: 3x2+xy)
- Trinomial, i.e. 3 terms (Example: 3x2+xy-6y2)
- Polynomial, i.e. more than 1 term (Binomial and trinomial examples are also polynomial)

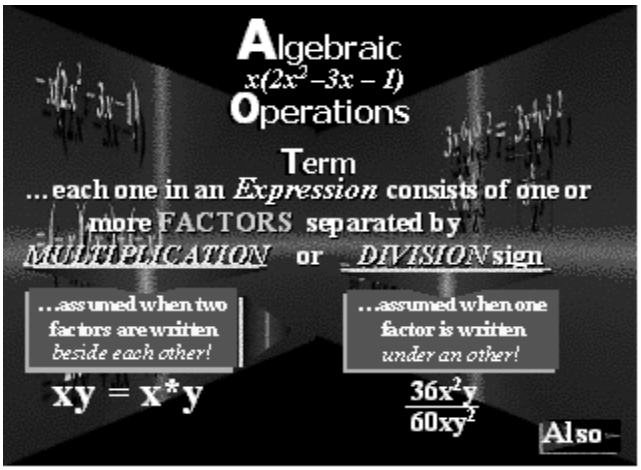
Algebraic operations in an expression consist of one or more FACTORs separated by MULTIPLICATION or DIVISION sign.

Multiplication is assumed when two factors are written beside each other.

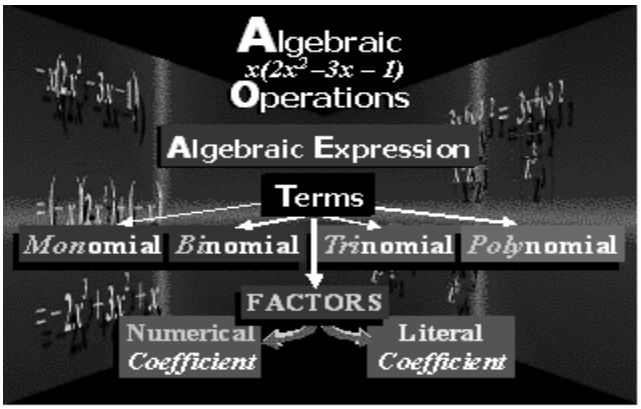
Example: xy = x*y

Division is assumed when one factor is written under an other.

Example: 36x2y / 60xy2



Factors can be further subdivided into NUMERICAL and LITERAL coefficients.

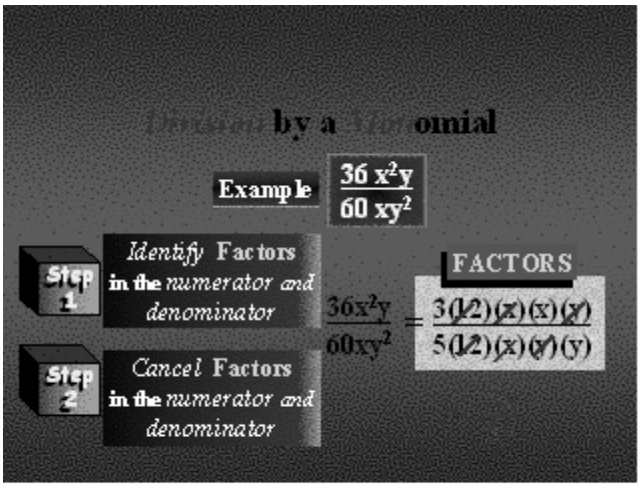


There are two steps for Division by a monomial.

1. Identify factors in the numerator and denominator
2. Cancel factors in the numerator and denominator

Example:

36x2y / 60xy2
36 can be factored as 3 x 12.
60 can be factored as 5 x 12
x2y can be factored as (x)(x)(y)
xy2 can be factored as (x)(y)(y)
Thus the expression is converted to: 3 x 12(x)(x)(y)/ 5 x 12(x)(y)(y)
12x(x)(y) in both numerator and denominator cancel each other. The result is: 3(x)/5(y)



Another example of division by a monomial is $(48a^2 – 32ab)/8a$.

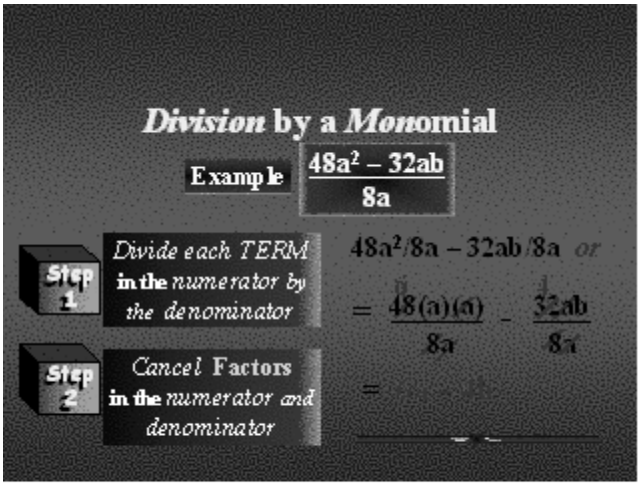
Here the steps are:

1. Divide each term in the numerator by the denominator
2. Cancel factors in the numerator and denominator

$48a^2 / 8a = 8x6(a)(a) / 8a = 6(a)$

$32(a)(b) / 8(a) = 4x8(a)(b) / 8(a) = 4(b)$

The answer is $6(a) – 4(b)$.

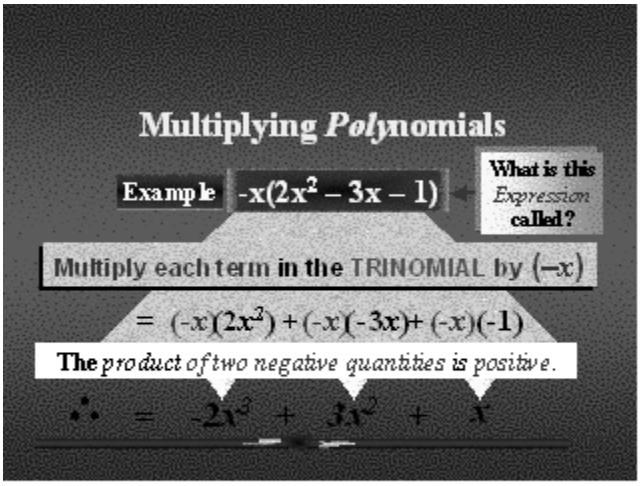


How to multiply polynomials? Look at the example –x(2x2 – 3x -1). Here each term in the trinomial 2x2 – 3x -1 is multiplied by –x.

= (-x)(2x2) + (-x)(-3x) + (-x)(-1)

= -2x3+ 3x2 + x

Please note that product of two negatives is positive.



(3x6y3 / x2z3)2

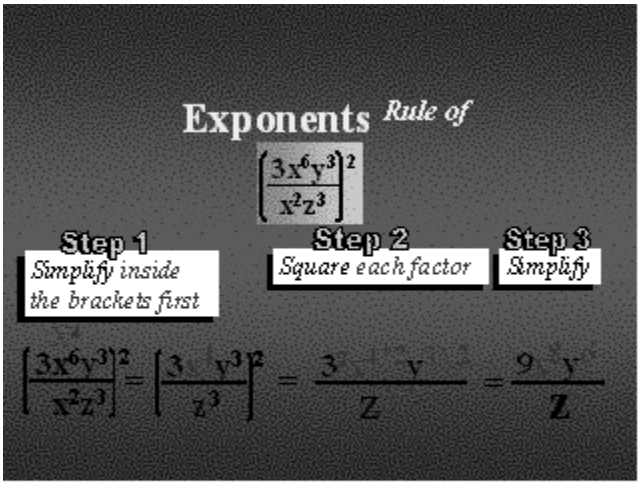
Exponent of a term means calculating some power of that term. In the example we are required to work out exponent of 3x6y3 / x2z3 to the power of 2. The steps in this calculation are:

1. Simplify inside the brackets first.
2. Square each factor
3. Simplify

In the first step, the expression 3x6y3 / x2z3 is first simplified to (3x4)(y3)/z3.

In the next step we take squares. The resulting expression is:

(32)(x4*2)(y3*2)/z3*2 = 9x8 y6 /z6

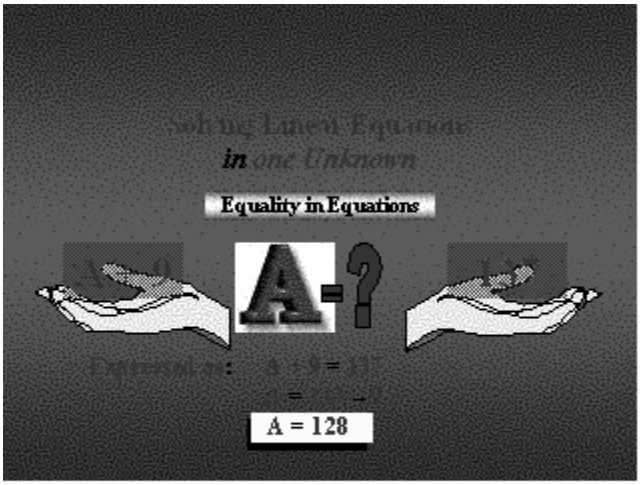


LINEAR EQUATION

If there is an expression A + 9 = 137, how do we calculate the value of A?

A = 137 – 9 = 128

As you see the term 9 was shifted to the right of the equality.



<u>To solve linear equations:</u>

1. Collect like terms
2. Divide both sides by numerical coefficient.

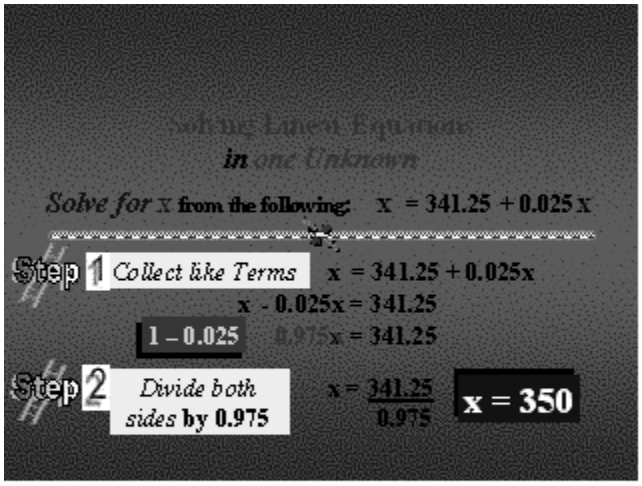
Step 1: x = 341.25 + 0.025x

x – 0.025x = 341.25

x(1-0.025) = 341.25

0.975x = 341.25

Step 2. x = 341.25/0.975 = 350


Solve Linear Equations
in one Unknown
Solve for x from the following: x = 341.25 + 0.025 x
Step 1 Collect like Terms
x = 341.25 + 0.025x
x - 0.025x = 341.25
1 – 0.025
0.975x = 341.25
Step 2 Divide both sides by 0.975
x = 341.25 / 0.975
x = 350

LECTURE # 19

Matrices

OBJECTIVES

The objectives of the lecture are to learn about:

- Matrices

QUESTIONS

Every student wonders why he or she should study matrices. There are manty important questions:
Where can we use Matrices?
Typical applications?
What is a Matrix?
What are Matrix operations?
Excel Matrix Functions?
There are many applications of matrices in business and industry especially where large amounts of data are processed daily.

TYPICAL APPLICATIONS

Practical questions in modern business and economic management can be answered with the help of matrix representation in:
Econometrics
Network Analysis
Decision Networks
Optimization
Linear Programming
Analysis of data
Computer graphics

WHAT IS A MATRIX?

A Matrix is a rectangular array of numbers. The plural of matrix is matrices. Matrices are usually represented with capital letters such as Matrix A, B, C.
For example

$$A=\begin{bmatrix}-1 & 9\\ -3 & 4\end{bmatrix} \quad B=\begin{bmatrix}47 & 62 & 70 & 56\\ 52 & 33 & 28 & 45\end{bmatrix} \quad C=\begin{bmatrix}1 & -4\\ 0 & 3\\ 5 & -2\end{bmatrix}$$

The numbers in a matrix are often arranged in a meaningful way. For example, the order for school clothing in September is illustrated in the table, as well as in the corresponding matrix.

	Size				
	Youth	S	M	L	XL
Sweat Pants	0	10	34	40	12
Sweat Shirts	18	25	29	21	7
Shorts	19	13	48	36	9
T-shirts	27	7	10	24	14

The data in the above table can be entered in the shape of a matrix as follows:

$$\begin{bmatrix} 0 & 10 & 34 & 40 & 12 \\ 18 & 25 & 29 & 21 & 7 \\ 19 & 13 & 48 & 36 & 9 \\ 27 & 7 & 10 & 24 & 14 \end{bmatrix}$$

DIMENSION

Dimension or Order of a Matrix = Number of Rows x Number of Columns

Example

Matrix T has dimensions of 2x3 or the order of matrix T is 2x3. '×' is just the notation, it do not mean to multiply both of them.

$$T = \begin{bmatrix} 6 & 2 & -1 \\ -4 & 0 & 7 \end{bmatrix} \begin{matrix} \leftarrow \text{row1} \\ \leftarrow \text{row2} \end{matrix}$$

$$\nearrow \quad \uparrow \quad \nwarrow$$

col 1 col 2 col 3

ROW, COLUMN AND SQUARE MATRIX

Suppose n = 1,2,3,4,…….

A matrix with dimensions 1xn is referred to as a row matrix

For example, matrix A to the right is a 1x4 row matrix.

A matrix with dimensions nx1 is referred to as a column matrix.

For example, matrix B in the middle is a 2x1 column matrix.

A matrix with dimensions nxn is referred to as a square matrix.

For example, matrix C is a 3x3 square matrix.

$$A = \begin{bmatrix} 12 & 17 & 10 & 9 \end{bmatrix} \quad B = \begin{bmatrix} -2 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -3 & 7 \\ 6 & 8 & -1 \\ -5 & 0 & 4 \end{bmatrix}$$

IDENTITY MATRIX

An identity matrix is a square matrix with 1's on the main diagonal from the upper left to the lower right and 0's off the main diagonal. An identity matrix is denoted as I. Some examples of identity matrices are shown below. The subscript indicates the size of the identity matrix. For example, I_n, represents an identity matrix with dimensions $n \times n$.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

MULTIPLICATIVE IDENTITY

With real numbers, the number 1 is referred to as a multiplicative identity because it has the unique property that the product a real number and 1 is that real number. In other words, 1 is called a multiplicative identity because for any real number n, $1 \bullet n = n$ and $n \bullet 1 = n$. With matrices, the identity matrix shares the same unique property as the number 1. In other words, a 2×2 identity matrix is a multiplicative inverse because for any 2×2 matrix A, $I_2 \bullet A = A$ and $A \bullet I_2 = A$

Example

Given the 2×2 matrix, $A = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$

$$I_2 \bullet A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

$$A \bullet I_2 = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

Work

r1c1 = 1(2) + 0(-3) = 2
r2c1 = 0(2) + 1(-3) = -3
r1c1 = 2(1) + -1(0) = 2
r2c1 = -3(1) + 4(0) = -3
r1c2 = 1(-1) + 0(4) = -1
r2c2 = 0(-1) + 1(4) = 4
r1c2 = 2(0) + -1(1) = -1
r2c2 = -3(0) + 4(1) = 4

where 'r' is for row and 'c' is for column.

LECTURE # 20

Operation on Matrices

OBJECTIVES

The objectives of the lecture are to learn about:

- **Matrices Operations**

EXAMPLE 1

An athletic clothing company manufactures T-shirts and sweat shirts in four differents sizes, small, medium, large, and x-large. The company supplies two major universities, the U of R and the U of S. The tables below show September's clothing order for each university

University of S's September Clothing Order

	S	M	L	XL
T-shirts	100	300	500	300
sweat shirts	150	400	450	250

University of R's September Clothing Order.

	S	M	L	XL
T-shirts	60	250	400	250
sweat shirts	100	200	350	200

Matrix Representation

The above information can be given by two matrices S and R as shown below.

$$S = \begin{bmatrix} 100 & 300 & 500 & 300 \\ 150 & 400 & 450 & 250 \end{bmatrix}$$

$$R = \begin{bmatrix} 60 & 250 & 400 & 250 \\ 100 & 200 & 350 & 220 \end{bmatrix}$$

MATRIX OPERATIONS

The matrix operations can be summarized as under:

- Organize and interpret data using matrices
- Use matrices in business applications
- Add and subtract two matrices
- Multiply a matrix by a scalar
- Multiply two matrices
- Interpret the meaning of the elements within a product matrix

PRODUCTION

The clothing company production in preparation for the universities' September orders is shown by the table and corresponding matrix P below.

	S	M	L	XL
T-shirts	300	700	900	500
sweat shirts	300	700	900	500

$$\mathbf{P} = \begin{bmatrix} 300 & 700 & 900 & 500 \\ 300 & 700 & 900 & 500 \end{bmatrix}$$

ADDITION AND SUBTRACTION OF MATRICES

The sum or difference of two matrices is calculated by adding or subtracting the corresponding elements of the matrices.
To add or subtract matrices, they must have the same dimensions.

PRODUCTION REQUIREMENT

Since the U of S ordered 100 small T-shirts and the U of R ordered 60, then althogether 160 small T-shirts are required to supply both universities. Thus, to calculate the total number of T-shirts and sweat shirts required to supply both universities, add the corresponding elements of the two order matrices as shown below.

$$\begin{bmatrix} 100 & 300 & 500 & 300 \\ 150 & 400 & 450 & 250 \end{bmatrix} + \begin{bmatrix} 60 & 250 & 400 & 250 \\ 100 & 200 & 350 & 220 \end{bmatrix}$$

$$= \begin{bmatrix} 160 & 550 & 900 & 550 \\ 250 & 600 & 800 & 470 \end{bmatrix}$$

OVERPRODUCTION
Since the company produced 300 small T-shirts and the received orders for only 160 small T-shirts, then the company produced 140 small T-shirts too many. Thus, to determine the company's over-production, subtract the corresponding elements of the total order matrix from the production matrix as shown below.

$$\begin{bmatrix} 300 & 700 & 900 & 500 \\ 300 & 700 & 900 & 500 \end{bmatrix} - \begin{bmatrix} 160 & 550 & 900 & 550 \\ 250 & 600 & 800 & 470 \end{bmatrix} = \begin{bmatrix} 140 & 150 & 0 & -50 \\ 50 & 100 & 100 & 30 \end{bmatrix}$$

MULTIPLY A MATRIX BY A SCALAR

Given a matrix *A* and a number *c*, the scalar multiplication *cA* is computed by multiplying the scalar *c* by every element of *A* .For example:

$$2\begin{bmatrix} 1 & 8 & -3 \\ 4 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 2 \times 1 & 2 \times 8 & 2 \times -3 \\ 2 \times 4 & 2 \times -2 & 2 \times 5 \end{bmatrix} = \begin{bmatrix} 2 & 16 & -6 \\ 8 & -4 & 10 \end{bmatrix}$$

MULTIPLICATION OF MATRICES

To understand the reasoning behind the definition of matrix multiplication, let us consider the following example.
Competing Companies, A and B, sell juice in 591 mL, 1 L and 2 L plastic bottles at prices of Rs.1.60, Rs.2.30 and Rs.3.10, respectively. The table below summarises the sales for the two companies during the month of July.

	591mL	1L	2L
Company A	20,000	5,500	10,600
Company B	18,250	7,000	11,000

What is total revenue of Company A?
What is total revenue of Company B?
Matrices may be used to illustrate the above information.
As shown at the right, the sales can be written as a 2X3 matrix, S, the selling prices can be written as a column matrix, P, and the total revenue for each company can be expressed as a column matrix, R.

$$S = \begin{bmatrix} 20,000 & 5,500 & 10,600 \\ 18,250 & 7,000 & 11,000 \end{bmatrix} \qquad P = \begin{bmatrix} 1.60 \\ 2.30 \\ 3.10 \end{bmatrix} \qquad R = \begin{bmatrix} 77,510 \\ 79,400 \end{bmatrix}$$

Since revenue is calculated by multiplying the number of sales by the selling price, the total revenue for each company is found by taking the product of the sales matrix and the price matrix.

$$\begin{bmatrix} 20,000 & 5,500 & 10,600 \\ 18,250 & 7,000 & 11,000 \end{bmatrix} \times \begin{bmatrix} 1.60 \\ 2.30 \\ 3.10 \end{bmatrix} = \begin{bmatrix} 77,510 \\ 79,400 \end{bmatrix}$$

Consider how the first row of matrix S and the single column P lead to the first entry of R.

$$\begin{bmatrix} 20,000 & 5,500 & 10,600 \\ 18,250 & 7,000 & 11,000 \end{bmatrix} \times \begin{bmatrix} 1.60 \\ 2.30 \\ 3.10 \end{bmatrix} \quad \Rightarrow 20,000(1.60) + 5,500(2.30) + 10,600(3.10) \Rightarrow \quad \begin{bmatrix} 77,510 \\ 79,400 \end{bmatrix}$$

Product of First Entries; Product of Second Entries; Product of Third Entries

With the above in mind, we define the product of a row and a column to be the number obtained by multiplying corresponding entries (first by first, second by second, and so on) and adding the results.

MULTIPLICATION RULES

If matrix A is a m×n matrix and matrix B is a n×p matrix, then the product AB is the m×p matrix whose entry in the i-th row and the j-th column is the product of the i-th row of matrix A and the j-th row of matrix B.
The product of a row and a column is the number obtained by multiplying corresponding elements (first by first, second by second, and so on).

To multiply matrices, the number of columns of A must equal the number of rows of B.

EXAMPLE:-
Given the matrices below, decide if the indicated product exists. And, if the product exists, determine the dimensions of the product matrix.

$$A=\begin{bmatrix}-5 & 0 & 4\\ 12 & -3 & 10\\ 7 & 15 & -1\end{bmatrix}$$

$$B=\begin{bmatrix}1 & 4\\ 9 & -1\\ -5 & 7\end{bmatrix}$$

$$C=\begin{bmatrix}4 & -3 & 1 & -7\\ 8 & -5 & -2 & 10\end{bmatrix}$$

$$D=\begin{bmatrix}1 & -1\\ 0 & 2\end{bmatrix}$$

$$E=\begin{bmatrix}6\\ 7\\ 8\\ 9\end{bmatrix}$$

MULTIPLICATION CHECKS
The table below gives a summary whether it is possible to multiply two matrices.
It may be noticed that the product of matrix A and matrix B is possible as the number of columns of A are equal to the number of rows of B. The product BA is not possible as the number of columns of b are not equal to rows of A.

Product	**Dimensions of the Matrices**	**Does a product exist?** (Is it possible to multiply the given matrices in this order?)	**Dimensions of Product Matrix**
AB	$A = 3\times 3 \quad B = 3\times 2$ (inner dimensions)	Yes, the product exists since the inner dimensions match (# of columns of *A* = # of rows of *B*).	3×2
BA	$B = 3\times 2 \quad A = 3\times 3$ (inner dimensions)	No, the product does not exist since the inner dimensions do not match (# of columns of *B* ≠ # of rows of *A*).	n/a

MULTIPLICATIVE

INVERSES

Real Numbers

Two non-zero real numbers are multiplicative inverses of each other if their products, in both orders, is 1. Thus,

the multiplicative inverse of a real number, x is $\frac{1}{x}$ or x^{-1} since $x \cdot \frac{1}{x} = 1$ and $\frac{1}{x} \cdot x = 1$.

Example:

The multiplicative inverse of 5 is $\frac{1}{5}$ since

$5 \cdot \frac{1}{5} = 1$ and $\frac{1}{5} \cdot 5 = 1$

Matrices

Two 2×2 matrices are inverses of each other if their products, in both orders, is a 2×2 identity matrix. Thus, the multiplicative inverse of a 2×2 matrix, A is A^{-1} since $A \cdot A^{-1} = I_2$ and $A^{-1} \cdot A = I_2$

Example

The multiplicative inverse of a matrix, $\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$ is $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ since is $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

$$\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

LECTURE # 21

Ratios and Proportions

ESTIMATING USING RATIOS-EXAMPLE 1

In the previous lecture, we studied how ratios can be used to determine unknowns. Here is another example with a slightly different approach. Here, ratios between the quantities.and data of only one quantity is known. We will estimate the total quantity that can be made? It is the quantity of orange juice that will determine the total quantity that can be made. Again the method is to use the ratio of the unknown to the known.

Punch recipe

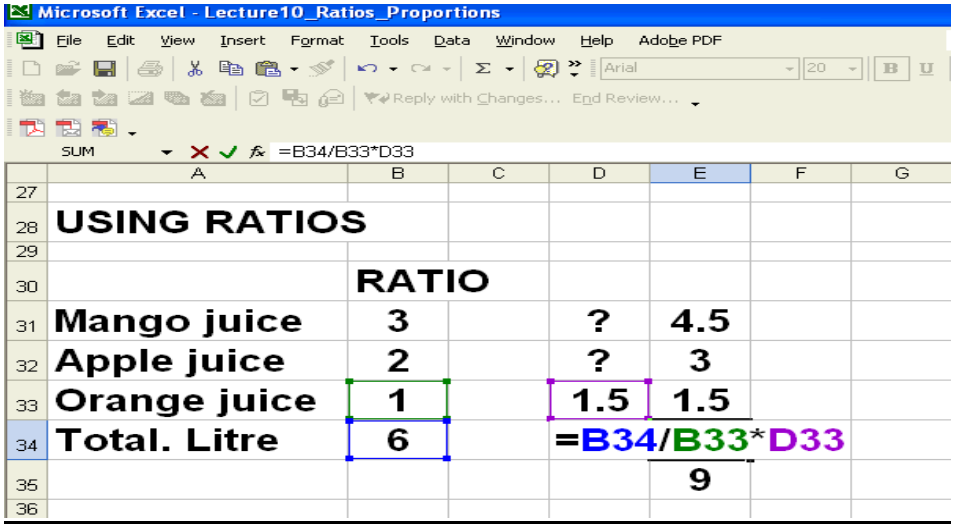
In a punch the ratio of mango juice, apple juice and orange juice is 3:2:1. If you have 1.5 liters of orange juice, how much punch can you make?

Calculation

Mango juice : Apple juice : Orange juice
3 : 2 : 1
Total = 3+2+1 = 6
X (say) : Y(say) : Z(say)
? : ? : 1.5litres
Total = ? litre
Mango juice (X) = (3/1)×1.5 = 4.5 litre
Apple juice (Y) = (2/1)×1.5 = 3.0 litre
Orange juice (Z) = 1.5 litre
Punch = 4.5 + 3.0 + 1.5 = 9 litres

EXCEL Calculation

The method used is the same as in previous examples.



ESTIMATING USING RATIOS-EXAMPLE 2

In a punch ratio of mango juice, apple juice and orange juice is 3 : 2 : 1.5. If you have 500 litres of orange juice find how much mango and apple juices are required to make the punch.

Punch recipe

The ratio of mango juice, apple juice and orange juice is 3 : 2 : 1.5 If you have 500 milliliters of orange juice, how much mango juice and apple juice is needed?

Mango juice : Apple juice : Orange juice

3 : 2 : 1.5

Total = 6.5

X (say) : Y(say) : Z (say)

? : ? : 500 litres

Total = ? litres

Mango juice (X) = (3/1.5)*500 = 1000 litre

Apple juice (Y) = (2/1.5)*500 = 667 litre

Orange juice (Z) = 500 litre

Punch = 1000 + 667 + 500 = 2167 litre

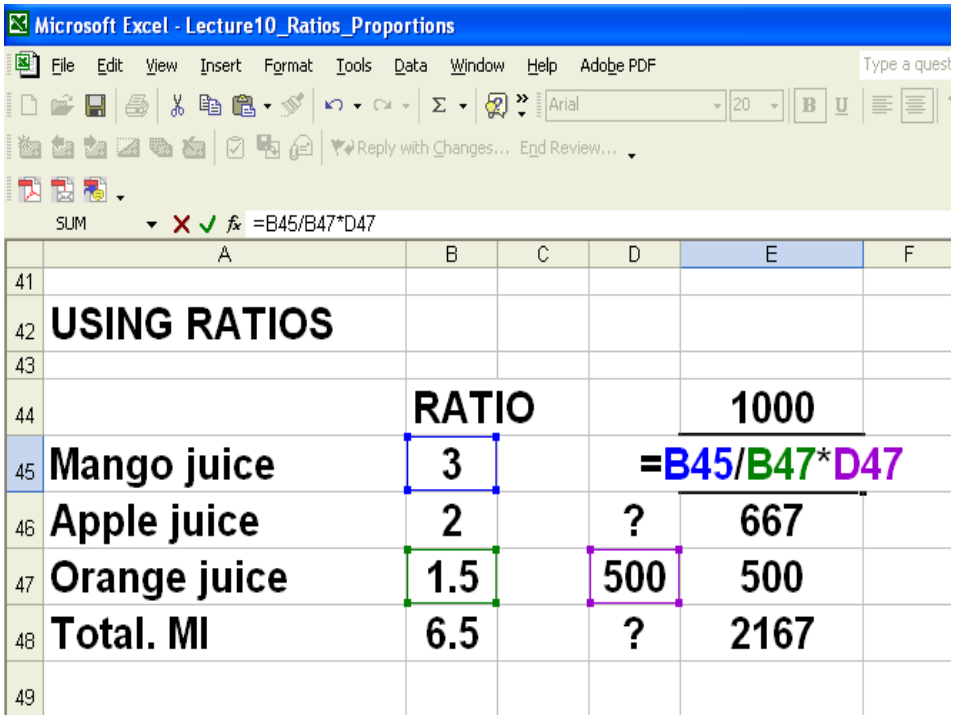
EXCEL Calculation

Here also ratios were used.

Mango juice = B45/B47*D47

Apple juice = B46/B47*D47

Orange juice = D47



EXAMPLE

In a certain class, the ratio of passing grades to failing grades is 7 to 5. How many of the 36 students failed the course?

The ratio, "7 to 5" (or 7 : 5 or 7/5), tells you that, of every 7 + 5 = 12 students, five failed. That is, 5/12 of the class failed.

Then (5/12)(36) = 15 students failed.

PROPORTION

a/b = c/d

...the values in the "b" and "c" positions are called the "means" of the proportion, while the values in the "a" and "d" positions are called the "extremes" of the proportion. A basic defining property of a proportion is that the product of the means is equal to the product of the extremes. In other words, given:

a/b = c/d

...it is a fact that ad = bc.

PROPORTION-EXAMPLES

Is 24/140 proportional to 30/176?

Check:

140×30=4200

24×176 = 4224

So the answer is that given ratios are not proportional.

PROPORTION EXAMPLE 1

Find the unknown value in the proportion: 2 : x = 3 : 9.

2 : x = 3 : 9

First, convert the colon-notation ratios to fractions:

2/x = 3/9

18=3x

x = 6

PROPORTION EXAMPLE 2

Find the unknown value in the proportion: $(2x + 1) : 2 = (x + 2) : 5$

$(2x + 1) : 2 = (x + 2) : 5$

First, convert the colon-notation ratios to fractions:

$(2x + 1)/2 = (x + 2)/5$

Then solve:

$5(2x + 1) = 2(x + 2)$

$10x + 5 = 2x + 4$

$8x = -1$

$x = -1/8$

MERCHANDISING

What does merchandising cover?

- Understand the ordinary dating notation for the terms of payment of an invoice.
- Solve merchandise pricing problems involving mark ups and markdowns.
- Calculate the net price of an item after single or multiple trade discounts.
- Calculate a single discount rate that is equivalent to a series of multiple discounts.
- Calculate the amount of the cash discount for which a payment qualifies.

STAKEHOLDERS IN Merchandising

Who are the stakeholders in merchandising?

The main players are:

- Manufacturer
- Middleman
- Retailer
- Consumer

There are discounts at all levels in the above chain.

MIDDLEMAN

A middle man is a person who buys a product directly from the manufacturer, and then either sells the product at retail prices to the public, or sells the product at wholesale prices to a distributor. There can often be more than one middle man when the latter practice is adopted. A middle man can purchase from the manufacturer and then work with another middle man who buys for the distributor. The manufacturer often views the middle man as the alternative to direct distribution.

List price or Retail price

List price refers to the manufacturer's suggested retail pricing. It may or may not be the price asked of the consumer. Much depends on

1. the product itself,
2. the built-in profit margin,
3. Supply and demand.

A product that is in high demand with low availability will sometimes sell *higher* than the list price, though this is less common than the reverse.

Virtually all products have a suggested retail or list price. Resellers (middleman, retailer) buy products in bulk and get a substantial discount in order to be able to get profit from selling the product at or below list price.

Trade Discount

Let L is the list price, then amount of trade discount is some percentage of this price. List price less amount of discount is the net price. In mathematical terms, we can write:

Amount of discount = $d \times L$

Where, d = Percentage of Discount
L = List Price
Net Price = L – Ld = L(1 – d)
Net Price = List Price – Amount of Discount

LECTURE # 22

Introduction to Statistics

WHAT IS STATISTICS?

- That science which enables us to draw conclusions about various phenomena on the basis of real data collected on sample-basis
- A tool for data-based research
- Also known as Quantitative Analysis
- A lot of application in a wide variety of disciplines … Agriculture, Anthropology, Astronomy, Biology, Economic, Engineering, Environment, Geology, Genetics, Medicine, Physics, Psychology, Sociology, Zoology …. Virtually every single subject from Anthropology to Zoology …. A to Z!
- Any scientific enquiry in which you would like to base your conclusions and decisions on real-life data, you need to employ statistical techniques!
- Now a day, in the developed countries of the world, there is an active movement for of Statistical Literacy.

MEANINGs OF 'STATISTICS':

The word "Statistics" which comes from the Latin words *status*, meaning a political state, originally meant information useful to the state, for example, information about the sizes of population sand armed forces. But this word has now acquired different meanings.

- In the *first place*, the word *statistics* refers to "numerical facts systematically arranged". In this sense, the word statistics is always used in plural. We have, for instance, statistics of prices, statistics of road accidents, statistics of crimes, statistics of births, statistics of educational institutions, etc. In all these examples, the word statistics denotes a set of numerical data in the respective fields. This is the meaning the man in the street gives to the word *Statistics* and most people usually use the word *data* instead.
- In the *second place*, the word *statistics* is defined as a discipline that includes procedures and techniques used to collect, process and analyze numerical data to make inferences and to research decisions in the face of uncertainty. It should of course be borne in mind that uncertainty does not imply ignorance but it refers to the incompleteness and the instability of data available. In this sense, the word statistics is used in the singular. As it embodies more of less all stages of the general process of learning, sometimes called *scientific method*, statistics is characterized as a science. Thus the word *statistics* used in the plural refers to a set of numerical information and in the singular, denotes the science of basing decision on numerical data. It should be noted that statistics as a subject is mathematical in character.
- *Thirdly*, the word statistics are numerical quantities calculated from sample observations; a single quantity that has been so collected is called a *statistic*. The mean of a sample for instance is a statistic. The word statistics is plural when used in this sense.

CHARACTERISTICS OF THE SCIENCE OF STATISTICS:

Statistics is a discipline in its own right. It would therefore be desirable to know the characteristic features of statistics in order to appreciate and understand its general nature. Some of its important characteristics are given below:

i) Statistics deals with the behaviour of aggregates or large groups of data. It has nothing to do with what is happening to a particular individual or object of the aggregate.

ii) Statistics deals with aggregates of observations of the same kind rather than isolated figures.

iii) Statistics deals with variability that obscures underlying patterns. No two objects in this universe are exactly alike. If they were, there would have been no statistical problem.

iv) Statistics deals with uncertainties as every process of getting observations whether controlled or uncontrolled, involves deficiencies or chance variation. That is why we have to talk in terms of probability.

v) Statistics deals with those characteristics or aspects of things which can be described numerically either by counts or by measurements.

vi) Statistics deals with those aggregates which are subject to a number of random causes, *e.g.* the heights of persons are subject to a number of causes such as race, ancestry, age, diet, habits, climate and so forth.

vii) Statistical laws are valid *on the average* or in the long run. There is n guarantee that a certain law will hold in all cases. Statistical inference is therefore made in the face of uncertainty.

viii) Statistical results might be misleading the incorrect if sufficient care in collecting, processing and interpreting the data is not exercised or if the statistical data are handled by a person who is not well versed in the subject mater of statistics.

THE WAY IN WHICH STATISTICS WORKS:

As it is such an important area of knowledge, it is definitely useful to have a fairly good idea about the way in which it works, and this is exactly the purpose of this introductory course.
The following points indicate some of the main functions of this science:

- Statistics assists in summarizing the larger set of data in a form that is easily understandable.
- Statistics assists in the efficient design of laboratory and field experiments as well as surveys.
- Statistics assists in a sound and effective planning in any field of inquiry.
- Statistics assists in drawing general conclusions and in making predictions of how much of a thing will happen under given conditions.

IMPORTANCE OF STATISTICS IN VARIOUS FIELDS

As stated earlier, Statistics is a discipline that has finds application in the most diverse fields of activity. It is perhaps a subject that should be used by everybody. Statistical techniques being powerful tools for analyzing numerical data are used in almost every branch of learning. In all areas, statistical techniques are being increasingly used, and are developing very rapidly.

- A modern administrator whether in public or private sector leans on statistical data to provide a factual basis for decision.

- A politician uses statistics advantageously to lend support and credence to his arguments while elucidating the problems he handles.
- A businessman, an industrial and a research worker all employ statistical methods in their work. Banks, Insurance companies and Government all have their statistics departments.

- A social scientist uses statistical methods in various areas of socio-economic life of a nation. It is sometimes said that "a social scientist without an adequate understanding of statistics, is often like the blind man groping in a dark room for a black cat that is not there".

The Meaning of Data:

The word "data" appears in many contexts and frequently is used in ordinary conversation. Although the word carries something of an aura of scientific mystique, its meaning is quite simple and mundane. It is Latin for "those that are given" (the singular form is "datum"). Data may therefore be thought of as the *results of observation*.

EXAMPLES OF DATA

Data are collected in many aspects of everyday life.

- Statements given to a police officer or physician or psychologist during an interview are data.

- So are the correct and incorrect answers given by a student on a final examination.

- Almost any athletic event produces data.

- The time required by a runner to complete a marathon,

- The number of errors committed by a baseball team in nine innings of play.

And, of course, data are obtained in the course of scientific inquiry:

- the positions of artifacts and fossils in an archaeological site,

- The number of interactions between two members of an animal colony during a period of observation,

- The spectral composition of light emitted by a star.

OBSERVATIONS AND VARIABLES:

Observation:

In statistics, an *observation* often means any sort of numerical recording of information, whether it is a physical measurement such as height or weight; a classification such as heads or tails, or an answer to a question such as yes or no.

Variables:

A characteristic that varies with an individual or an object, is called a *variable*. For example, age is a variable as it varies from person to person. A variable can assume a number of values. The given set of all possible values from which the variable takes on a value is called its Domain. If for a given

problem, the domain of a variable contains only one value, then the variable is referred to as a *constant*.

QUANTITATIVE AND QUALITATIVE VARIABLES:

Variables may be classified into quantitative and qualitative according to the form of the characteristic of interest.

A variable is called a *quantitative variable* when a characteristic can be expressed numerically such as age, weight, income or number of children. On the other hand, if the characteristic is non-numerical such as education, sex, eye-colour, quality, intelligence, poverty, satisfaction, etc. the variable is referred to as a *qualitative variable*. A qualitative characteristic is also called an *attribute*. An individual or an object with such a characteristic can be counted or enumerated after having been assigned to one of the several mutually exclusive classes or categories.

Discrete and Continuous Variables:

A quantitative variable may be classified as discrete or continuous. *A discrete* variable is one that can take only a discrete set of integers or whole numbers, which is the values are taken by jumps or breaks. A discrete variable represents *count* data such as the number of persons in a family, the number of rooms in a house, the number of deaths in an accident, the income of an individual, etc.

A variable is called a *continuous* variable if it can take on any value-fractional or integral—within a given interval, ***i.e.*** its domain is an interval with all possible values without gaps. A continuous variable represents measurement data such as the age of a person, the height of a plant, the weight of a commodity, the temperature at a place, etc.

A variable whether countable or measurable, is generally denoted by some symbol such as X or Y and X_i or X_j represents the ith or j^{th} value of the variable. The subscript i or j is replaced by a number such as 1,2,3, … when referred to a particular value.

Measurement Scales:

By *measurement*, we usually mean the assigning of number to observations or objects and scaling is a process of measuring. The four scales of measurements are briefly mentioned below:

NOMINAL SCALE:

The classification or grouping of the observations into mutually exclusive qualitative categories or classes is said to constitute a *nominal scale*. For example, students are classified as male and female. Number 1 and 2 may also be used to identify these two categories. Similarly, rainfall may be classified as heavy moderate and light. We may use number 1, 2 and 3 to denote the three classes of rainfall. The numbers when they are used only to identify the categories of the given scale, carry no numerical significance and there is no particular order for the grouping.

ORDINAL OR RANKING SCALE:

It includes the characteristic of a nominal scale and in addition has the property of *ordering* or *ranking* of measurements. For example, the performance of students (or players) is rated as excellent, good fair or poor, etc. Number 1, 2, 3, 4 etc. are also used to indicate ranks. The only relation that holds between any pair of categories is that of "greater than" (or more preferred).

INTERVAL SCALE:

A measurement scale possessing a constant interval size (distance) but not a true zero point, is called an *interval scale*. Temperature measured on either the Celsius or the Fahrenheit scale is an outstanding example of interval scale because the same difference exists between 20^o C (68^o F) and 30^o C (86^o F) as between 5^o C (41^o F) and 15^o C (59^o F). It cannot be said that a temperature of 40 degrees is twice as hot as a temperature of 20 degree, i.e. the ratio 40/20 has no meaning. The arithmetic operation of addition, subtraction, etc. is meaningful.

RATIO SCALE:

It is a special kind of an interval scale where the sale of measurement has a true *zero* point as its origin. The ratio scale is used to measure weight, volume, distance, money, etc. The, key to differentiating interval and ratio scale is that the zero point is meaningful for ratio scale.

ERRORS OF MEASUREMENT:

Experience has shown that a continuous variable can never be measured with perfect fineness because of certain habits and practices, methods of measurements, instruments used, etc. the

measurements are thus always recorded correct to the nearest units and hence are of limited accuracy. The *actual* or *true* values are, however, assumed to exist. For example, if a student's weight is recorded as 60 kg (correct to the nearest kilogram), his true weight in fact lies between 59.5 kg and 60.5 kg, whereas a weight recorded as 60.00 kg means the true weight is known to lie between 59.995 and 60.005 kg. Thus there is a difference, however small it may be between the measured value and the true value. This sort of departure from the true value is technically known as the *error of measurement*. In other words, if the observed value and the true value of a variable are denoted by x and $x + \varepsilon$ respectively, then the difference $(x + \varepsilon) - x$, i.e. ε is the error. This error involves the unit of measurement of x and is therefore called an *absolute error*. An absolute error divided by the true value is called the *relative error*. Thus the relative error $= \dfrac{\varepsilon}{x + \varepsilon}$, which when multiplied by 100, is *percentage error*. These errors are independent of the units of measurement of x. It ought to be noted that an error has both magnitude and direction and that the word *error* in statistics does not mean mistake which is a chance inaccuracy.

BIASED AND RANDOM ERRORS:

An error is said to be *biased* when the observed value is consistently and constantly higher or lower than the true value. Biased errors arise from the personal limitations of the observer, the imperfection in the instruments used or some other conditions which control the measurements. These errors are not revealed by repeating the measurements. They are cumulative in nature, that is, the greater the number of measurements, the greater would be the magnitude of error. They are thus more troublesome. These errors are also called *cumulative* or *systematic errors*.

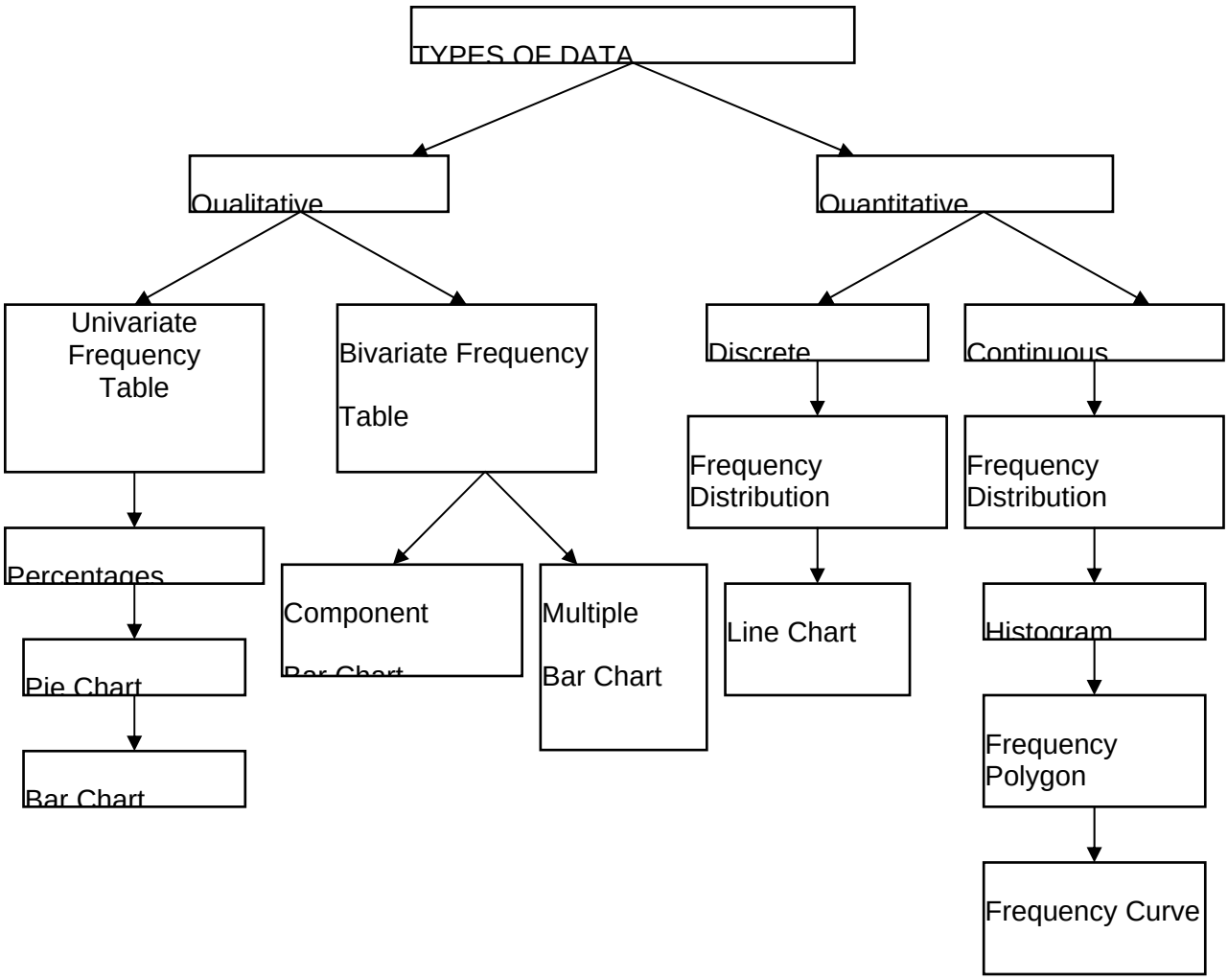
An error, on the other hand, is said to be unbiased when the deviations, i.e. the excesses and defects, from the true value tend to occur equally often. Unbiased errors and revealed when measurements are repeated and they tend to cancel out in the long run. These errors are therefore *compensating* and are also known as *random errors* or *accidental errors*.

LECTURE # 23

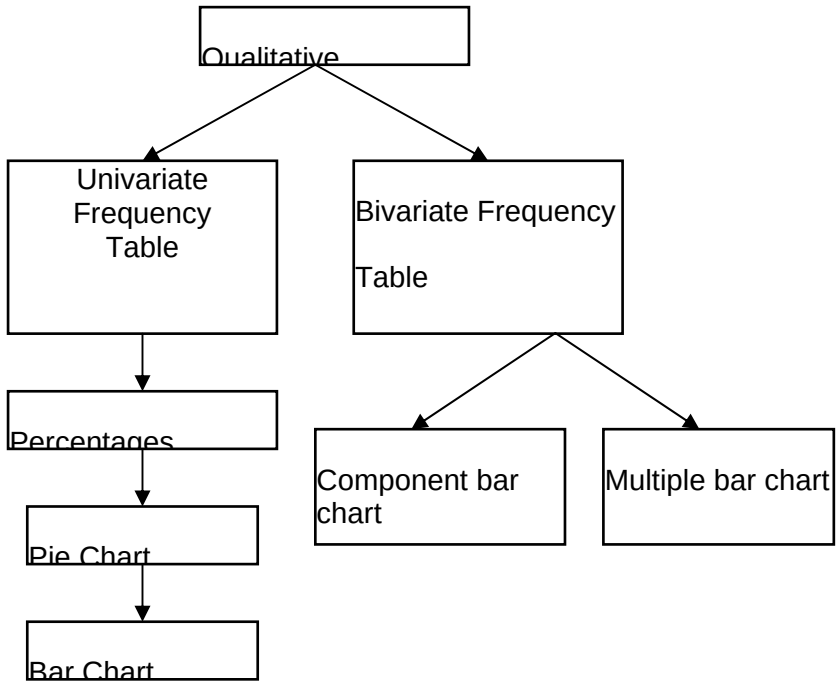
Data Representation

- Tabulation
- Simple bar chart
- Component bar chart
- Multiple bar chart
- Pie chart

As indicated in the last lecture, there are two broad categories of data … qualitative data and quantitative data. A variety of methods exist for summarizing and describing these two types of data. The tree-diagram below presents an outline of the various techniques



In today's lecture, we will be dealing with various techniques for summarizing and describing qualitative data.



We will begin with the univariate situation, and will proceed to the bivariate situation.

EXAMPLE:

Suppose that we are carrying out a survey of the students of first year studying in a co-educational college of Lahore. Suppose that in all there are 1200 students of first year in this large college.

We wish to determine what proportion of these students have come from Urdu medium schools and what proportion has come from English medium schools.

So we will interview the students and we will inquire from each one of them about their schooling. As a result, we will obtain a set of data as you can now see on the screen.

We will have an array of observations as follows:

U, U, E, U, E, E, E, U, ……

(U : URDU MEDIUM)
(E : ENGLISH MEDIUM)

Now, the question is what should we do with this data?

Obviously, the first thing that comes to mind is to count the number of students who said "Urdu medium" as well as the number of students who said "English medium".

This will result in the following table:

Medium of Institution	No. of Students (f)
Urdu	719
English	481
	1200

The technical term for the numbers given in the second column of this table is "frequency".

It means "how frequently something happens?"

Out of the 1200 students, 719 stated that they had come from Urdu medium schools.

So in this example, the frequency of the first category of responses is 719 whereas the frequency of the second category of responses is 481.

It is evident that this information is not as useful as if we compute the proportion or percentage of students falling in each category.

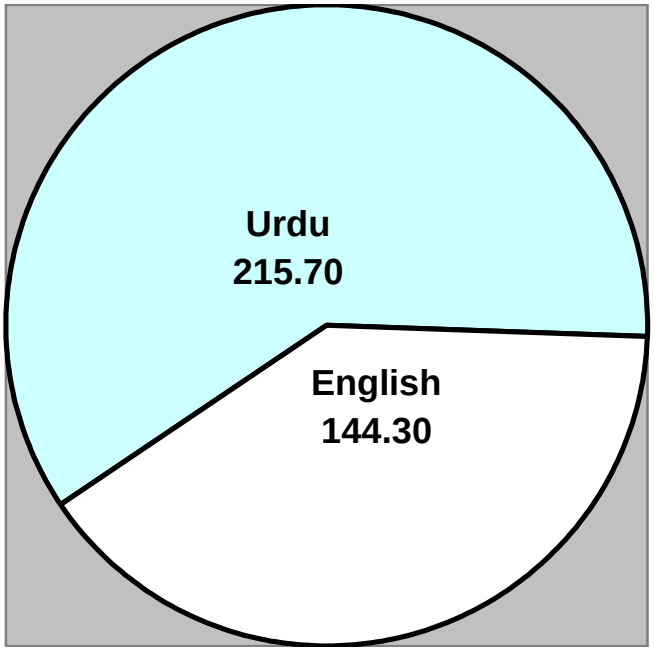
Dividing the cell frequencies by the total frequency and multiplying by 100 we obtain the following:

Medium of Institution	**f**	**%**
Urdu	719	59.9 = 60%
English	481	40.1 = 40%
	1200	

What we have just accomplished is an example of a univariate frequency table pertaining to qualitative data.
Let us now see how we can represent this information in the form of a diagram.
One good way of representing the above information is in the form of a pie chart.
A pie chart consists of a circle which is divided into two or more parts in accordance with the number of distinct categories that we have in our data.
For the example that we have just considered, the circle is divided into two sectors, the larger sector pertaining to students coming from Urdu medium schools and the smaller sector pertaining to students coming from English medium schools.
How do we decide where to cut the circle?
The answer is very simple! All we have to do is to divide the cell frequency by the total frequency and multiply by 360.
This process will give us the exact value of the angle at which we should cut the circle.

PIE CHART

Medium of Institution	**f**	**Angle**
Urdu	719	215.7^0
ENGLISH	481	144.3^0
	1200	


Urdu
215.70
English
144.30

SIMPLE BAR CHART:

The next diagram to be considered is the simple bar chart.

A simple bar chart consists of horizontal or vertical bars of equal width and lengths proportional to values they represent.

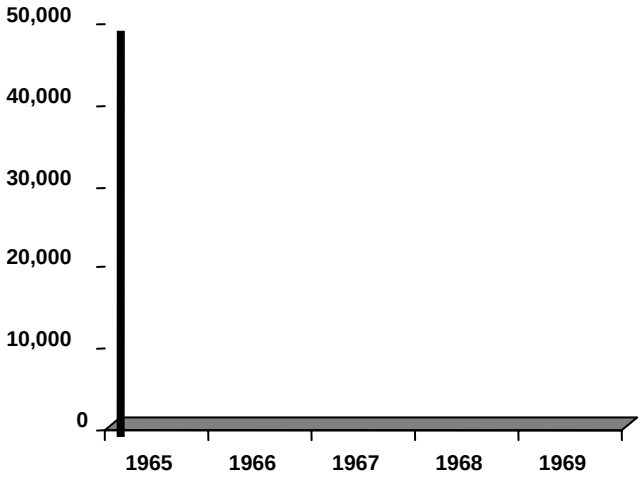
As the basis of comparison is one-dimensional, the widths of these bars have no mathematical significance but are taken in order to make the chart look attractive.
Let us consider an example.

Suppose we have available to us information regarding the turnover of a company for 5 years as given in the table below:

Years	1965	1966	1967	1968	1969
Turnover (Rupees)	35,000	42,000	43,500	48,000	48,500

In order to represent the above information in the form of a bar chart, all we have to do is to take the year along the x-axis and construct a scale for turnover along the y-axis.

Next, against each year, we will draw vertical bars of equal width and different heights in

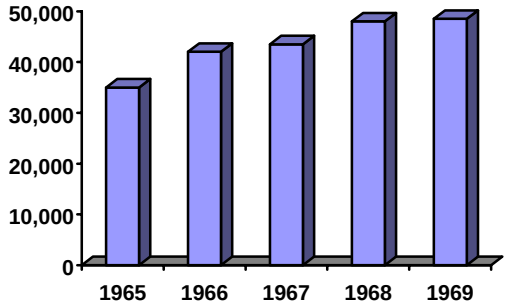


accordance with the
turn-over figures that we have in our table.

As a result we obtain a simple and attractive diagram as shown below.
When our values do not relate to time, they should be arranged in ascending or descending order before-charting.

BIVARIATE FREQUENCY TABLE:

What we have just considered was the univariate situation.
In each of the two examples, we were dealing with one single variable.
In the example of the first year students of a college, our lone variable of interest was 'medium of schooling'.



And in the second example, our one single variable of interest was turnover.Now let us expand the discussion a little, and consider the bivariate situation. Going back to the example of the first year students, suppose that alongwith the enquiry about the Medium of Institution, you are also recording the sex of the student.
Suppose that our survey results in the following information:

Student No.	**Medium**	**Gender**
1	U	F
2	U	M
3	E	M
4	U	F
5	E	M
6	E	F
7	U	M
8	E	M
:	:	:
:	:	:

Now this is a bivariate situation; we have two variables, medium of schooling and sex of the student. In order to summarize the above information, we will construct a table containing a box head and a stub as shown below:

Sex **Med.**	MALE	**Female**	**Total**
Urdu			
English			
Total			

The top row of this kind of a table is known as the boxhead and the first column of the table is known as stub.
Next, we will count the number of students falling in each of the following four categories:

1. Male student coming from an Urdu medium school.
2. Female student coming from an Urdu medium school.
3. Male student coming from an English medium school.
4. Female student coming from an English medium school.

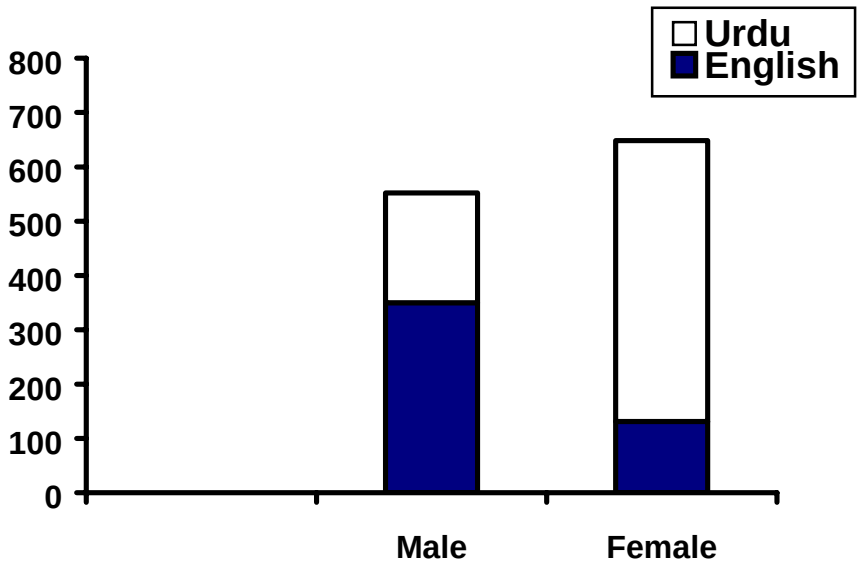
As a result, suppose we obtain the following figures:

Sex **Med.**	MALE	**Female**	**Total**
Urdu	202	517	719
English	350	131	481
Total	552	648	1200

What we have just accomplished is an example of a bivariate frequency table pertaining to two qualitative variables.

COMPONENT BAR CHAR:

Let us now consider how we will depict the above information diagrammatically.
This can be accomplished by constructing the component bar chart (also known as the subdivided bar chart) as shown below:



In the above figure, each bar has been divided into two parts. The first bar represents the total number of male students whereas the second bar represents the total number of female students.

As far as the medium of schooling is concerned, the lower part of each bar represents the students coming from English medium schools. Whereas the upper part of each bar represents the students coming from the Urdu medium schools.The advantage of this kind of a diagram is that we are able to ascertain the situation of both the variables at a glance.

We can compare the number of male students in the college with the number of female students, and at the same time we can compare the number of English medium students among the males with the number of English medium students among the females.

MULTIPLE BAR CHART

The next diagram to be considered is the multiple bar chart.

Let us consider an example.

Suppose we have information regarding the imports and exports of Pakistan for the years 1970-71 to 1974-75 as shown in the table below:

Years	**Imports (Crores of Rs.)**	**Exports (Crores of Rs.)**
1970-71	370	200
1971-72	350	337
1972-73	840	855
1973-74	1438	1016
1974-75	2092	1029

Source: State Bank of Pakistan

A multiple bar chart is a very useful and effective way of presenting this kind of information.

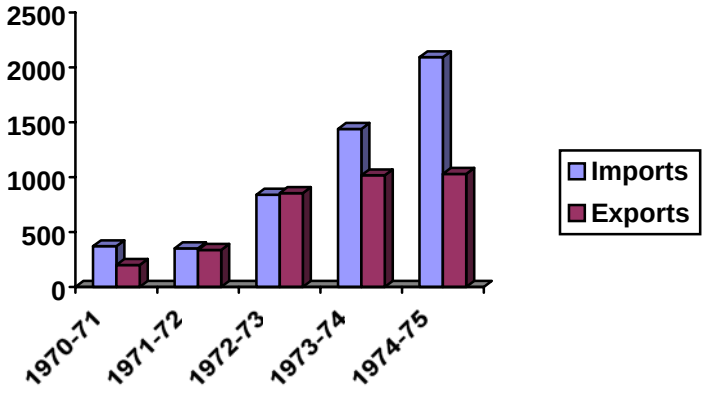
This kind of a chart consists of a set of grouped bars, the lengths of which are proportionate to the values of our variables, and each of which is shaded or coloured differently in order to aid identification.

With reference to the above example, we obtain the multiple bar chart shown below:

Multiple Bar Chart Showing

Imports & Exports
of Pakistan 1970-71 to 1974-75

This is a very good device for the comparison of two different kinds of information.



If, in addition to information regarding imports and exports, we also had information regarding production, we could

have compared them from year to year by grouping the three bars together.
The question is, what is the basic difference between a component bar chart and a multiple bar chart?
The component bar chart should be used when we have available to us information regarding totals and their components.

For example, the total number of male students out of which some are Urdu medium and some are English medium. The number of Urdu medium male students and the number of English medium male students add up to give us the total number of male students.
On the contrary, in the example of exports and imports, the imports and exports do not add up to give us the totality of some one thing!

LECTURE # 24

The Central Tendency of a data set

In today's lecture, we will continue with the concept of the mode, and will discuss the non-modal situation as well as the bi-modal situation.First of all, let us revise the discussion carried out at the end of the last lecture.You will recall that we picked up the example of the EPA mileage ratings, ancomputed the mode of this distribution by applying the following formula:

Mode:

$$\hat{X}=l+\frac{f_m-f_1}{(f_m-f_1)+(f_m-f_2)}xh$$

Where

l = lower class boundary of the modal class,

f_m = frequency of the modal class,

f_1 = frequency of the class preceding the modal class,

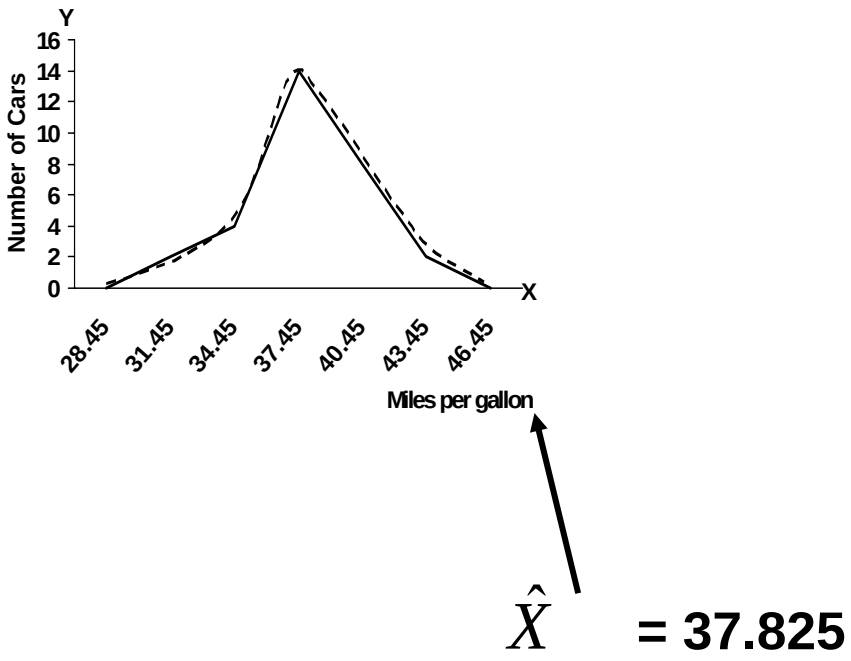
f_2 = frequency of the class following modal class, and

h = length of class interval of the modal class

Hence, we obtained:

$$\begin{aligned}\hat{X} &= 35.95+\frac{14-4}{(14-4)+(14-8)}\times 3\\ &= 35.95+\frac{10}{10+6}\times 3\\ &= 35.95+1.875\\ &= 37.825\end{aligned}$$

Subsequently, we considered the location of the mode with reference to the graphical picture of our frequency distribution.



In general, it was noted that, for most of the frequency distributions, the mode lies somewhere in the middle of our frequency distribution, and hence is eligible to be called a measure of central tendency. The mode has some very desirable properties.

DESIRABLE PROPERTIES OF THE MODE:

- The mode is easily understood and easily ascertained in case of a discrete frequency distribution.
- It is not affected by a few very high or low values.

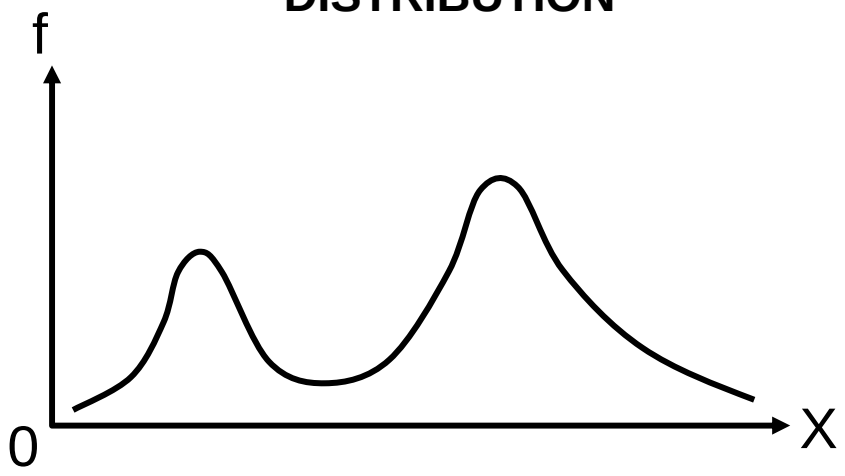
The question arises, "When should we use the mode?"

The answer to this question is that the mode is a valuable concept in certain situations such as the one described below:

Suppose the manager of a men's clothing store is asked about the average size of hats sold. He will probably think not of the arithmetic or geometric mean size, or indeed the median size. Instead, he will in all likelihood quote that particular size which is sold ***most often*.** This average is of far more use to him as a businessman than the arithmetic mean, geometric mean or the median. The modal size of all clothing is the size which the businessman must stock in the greatest quantity and variety in comparison with other sizes. Indeed, in most inventory (stock level) problems, one needs the mode more often than any other measure of central tendency. It should be noted that in some situations there may be no mode in a simple series where no value occurs more than once.

On the other hand, sometimes a frequency distribution contains two modes in which case it is called a bi-modal distribution as shown below:

THE BI-MODAL FREQUENCY DISTRIBUTION



The next measure of central tendency to be discussed is the arithmetic mean.

THE ARITHMETIC MEAN

The arithmetic mean is the statistician's term for what the layman knows as the average. It can be thought of as that value of the variable series which is *numerically* MOST representative of the whole series. Certainly, this is the most widely used average in statistics. Easiest In addition, it is probably the to calculate.

Its formal definition is:

ARITHMETIC MEAN:

The arithmetic mean or simply the mean is a value obtained by dividing the sum of all the observations by their number.

Arithmetic Mean:

$$\overline{X} = \frac{\textbf{Sum of all the observations}}{\textbf{Number of the observations}}$$

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

Where n represents the number of observations in the sample that has been the ith observation in the sample (i = 1, 2, 3, …, n), and represents the mean of the sample.

For simplicity, the above formula can be written as

$$\overline{X} = \frac{\sum X}{n}$$

In other words, it is not necessary to insert the subscript 'i'.)

EXAMPLE:
Information regarding the receipts of a news agent for seven days of a particular week are given below

Day	Receipt of News Agent
Monday	£ 9.90
Tuesday	£ 7.75
Wednesday	£ 19.50
Thursday	£ 32.75
Friday	£ 63.75
Saturday	£ 75.50
Sunday	£ 50.70
Week Total	£ 259.85

Mean sales per day in this week:

= £ 259.85/7 = £ 37.12
(To the nearest penny).
Interpretation:
The mean, £ 37.12, represents the amount (in pounds sterling) that would have been obtained on each day if the same amount were to be obtained on each day. The above example pertained to the computation of the arithmetic mean in case of ungrouped data i.e. raw data.

Let us now consider the case of data that has been grouped into a frequency distribution. When data pertaining to a continuous variable has been grouped into a frequency distribution, the frequency distribution is used to calculate the approximate values of descriptive measures --- as the identity of the observations is lost.
To calculate the approximate value of the mean, the observations in each class are assumed to be identical with the class midpoint Xi.
The mid-point of every class is known as its class-mark.
In other words, the midpoint of a class 'marks' that class.As was just mentioned, the observations in each class are assumed to be identical with the midpoint i.e. the class-mark. (This is based on the assumption that the observations in the group are evenly scattered between the two extremes of the class interval).
As was just mentioned, the observations in each class are assumed to be identical with the midpoint i.e. the class-mark.(This is based on the assumption that the observations in the group are evenly scattered between the two extremes of the class interval).

FREQUENCY DISTRIBUTION

Mid Point X	Frequency f
X_1	f_1
X_2	f_2
X_3	f_3
:	:
:	:
:	:
X_k	f_k

In case of a frequency distribution, the arithmetic mean is defined as:

ARITHMETIC MEAN

$$\overline{X} = \frac{\sum_{i=1}^{k} f_i X_i}{\sum_{i=1}^{k} f_i} = \frac{\sum_{i=1}^{k} f_i X_i}{n}$$

For simplicity, the above formula can be written as

$$\overline{X} = \frac{\sum fX}{\sum f} = \frac{\sum fX}{n}$$

(The subscript 'i' can be dropped.)

Let us understand this point with the help of an example:

Going back to the example of EPA mileage ratings that we dealt with when discussing the formation of a frequency distribution. The frequency distribution that we obtained was:

EPA MILEAGE RATINGS OF 30 CARS OF A CERTAIN MODEL

Class (Mileage Rating)	Frequency (No. of Cars)
30.0 – 32.9	2
33.0 – 35.9	4
36.0 – 38.9	14
39.0 – 41.9	8
42.0 – 44.9	2
Total	30

The first step is to compute the mid-point of every class.

(You will recall that the concept of the mid-point has already been discussed in an earlier lecture.)

CLASS-MARK (MID-POINT):

The mid-point of each class is obtained by adding the sum of the two limits of the class and dividing by 2.

Hence, in this example, our mid-points are computed in this manner:

30.0 plus 32.9 divided by 2 is equal to 31.45,

33.0 plus 35.9 divided by 2 is equal to 34.45,

And so on.

Class (Mileage Rating)	Class-mark (Midpoint) X
30.0 – 32.9	31.45
33.0 – 35.9	34.45
36.0 – 38.9	37.45
39.0 – 41.9	40.45
42.0 – 44.9	43.45

In order to compute the arithmetic mean, we first need to construct the column of fX, as shown below:

Class-mark (Midpoint) X	Frequency f	fX
31.45	2	62.9
34.45	4	137.8
37.45	14	524.3
40.45	8	323.6
43.45	2	86.9
	30	1135.5

Applying the formula

$$\overline{X} = \frac{\sum fX}{\sum f},$$

We obtain

$$\overline{X} = \frac{1135.5}{30} = 37.85$$

INTERPRETATION:

The average mileage rating of the 30 cars tested by the Environmental Protection Agency is 37.85 – on the average, these cars run 37.85 miles per gallon. An important concept to be discussed at this point is the concept of grouping error.

GROUPING ERROR:

"Grouping error" refers to the error that is introduced by the assumption that all the values falling in a class are equal to the mid-point of the class interval. In reality, it is highly improbable to have a class for which all the values lying in that class are equal to the mid-point of that class. This is why the mean that we calculate from a frequency distribution does not give exactly the same answer as what we would get by computing the mean of our raw data.

As indicated earlier, a frequency distribution is used to calculate the approximate values of various descriptive measures.(The word 'approximate' is being used because of the grouping error that was just discussed.) This grouping error arises in the computation of many descriptive measures such as the geometric mean, harmonic mean, mean deviation and standard deviation. But, experience has shown that in the calculation of the arithmetic mean, this error is usually small and never serious. Only a slight difference occurs between the true answer that we would get from the raw data, and the answer that we get from the data that has been grouped in the form of a frequency distribution.

In this example, if we calculate the arithmetic mean directly from the 30 EPA mileage ratings, we obtain:

Arithmetic mean computed from raw data of the EPA mileage ratings:

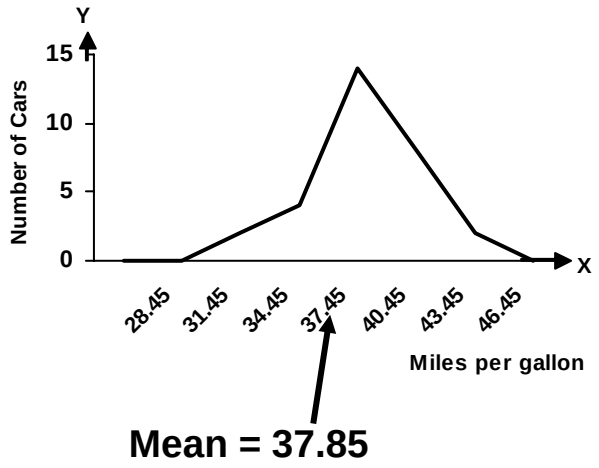
$$\bar{X} = \frac{363+301+....+339+398}{30}$$

$$= \frac{1134.7}{30} = 37.82$$

The difference between the true value of i.e. 37.82 and the value obtained from the frequency distribution i.e. 37.85 is indeed very slight. The arithmetic mean is predominantly used as a measure of central tendency.

The question is, "Why is it that the arithmetic mean is known as a measure of central tendency?"

The answer to this question is that we have just obtained i.e. 37.85 falls more or less in the centre of our frequency distribution.



As indicated earlier, the arithmetic mean is predominantly used as a measure of central tendency.

It has many desirable properties:

DESIRABLE PROPERTIES OF THE ARITHMETIC MEAN

- Best understood average in statistics.
- Relatively easy to calculate
- Takes into account every value in the series.

But there is one limitation to the use of the arithmetic mean:

As we are aware, every value in a data-set is included in the calculation of the mean, whether the value be high or low. Where there are a few very high or very low values in the series, their effect can be to *drag* the arithmetic mean towards them. this may make the mean unrepresentative.

Example:

Example of the Case Where the Arithmetic Mean Is Not a Proper Representative of the Data:

Suppose one walks down the main street of a large city centre and counts the number of floors in each building.

Suppose, the following answers are obtained:

5, 4, 3, 4, 5, 4, 3, 4, 5,

20, 5, 6, 32, 8, 27

The mean number of floors is 9 even though 12 out of 15 of the buildings have 6 floors or less.

The three skyscraper blocks are having a disproportionate effect on the arithmetic mean.

(Some other average in this case would be more representative.)
The concept that we just considered was the concept of the simple arithmetic mean.
Let us now discuss the concept of the weighted arithmetic mean.
Consider the following example:

EXAMPLE:

Suppose that in a particular high school, there are:-

100 – freshmen
80 – sophomores
70 – juniors
50 – seniors

And suppose that on a given day, 15% of freshmen, 5% of sophomores, 10% of juniors, 2% of seniors are *absent.*

The problem is that: What percentage of students is absent for the school as a whole on that particular day?
Now a student is likely to attempt to find the answer by adding the percentages and dividing by 4 i.e.

$$\frac{15+5+10+2}{4}=\frac{32}{4}=8$$

But the fact of the matter is that the above calculation gives a wrong answer.In order to figure out why this is a wrong calculation, consider the following:As we have already noted, 15% of the freshmen are absent on this particular day. Since, in all, there are 100 freshmen in the school, hence the total number of freshmen who are absent is also 15.

But as far as the sophomores are concerned, the total number of them in the school is 80, and if 5% of them are absent on this particular day, this means that the total number of sophomores who are absent is only 4.
Proceeding in this manner, we obtain the following table.

Category of Student	Number of Students in the school	Number of Students who are absent
Freshman	100	15
Sophomore	80	4
Junior	70	7
Senior	50	1
TOTAL	300	27

Dividing the total number of students who are absent by the total number of students enrolled in the school, and multiplying by 100, we obtain:

$$\frac{27}{300}\times 100 \quad = \quad 9$$

Thus its very clear that previous result was not correct.
This situation leads us to a very important observation, i.e. here our figures pertaining to absenteeism in various categories of students cannot be regarded as having equal weightage.

When we have such a situation, the concept of "weighing" applies i.e. every data value in the data set is assigned a certain weight according to a suitable criterion. In this way, we will have a weighted

series of data instead of an un-weighted one. In this example, the number of students enrolled in each category acts as the weight for the number of absences pertaining to that category i.e.

Category of Student	Percentage of Students who are absent X_i	Number of students enrolled in the school (Weights) W_i	W_iX_i (Weighted X_i)
Freshman	15	100	$100 \times 15 = 1500$
Sophomore	5	80	$80 \times 5 = 400$
Junior	10	70	$70 \times 10 = 700$
Senior	2	50	$50 \times 2 = 100$
	Total	$\Sigma W_i = 300$	$\Sigma W_iX_i = 2700$

The formula for the weighted arithmetic mean is:

WEIGHTED MEAN

$$\overline{X}_w = \frac{\sum W_i X_i}{\sum W_i}$$

And, in this example, the weighted mean is equal to:

$$\overline{X}_w = \frac{\sum W_i X_i}{\sum W_i}$$

$$= \frac{2700}{300}$$

$$= 9$$

Thus we note that, in this example, the weighted mean yields exactly the same as the answer that we obtained earlier.

As obvious, the weighing process leads us to a correct answer under the situation where we have data that cannot be regarded as being such that each value should be given equal weightage.

An important point to note here is the *criterion* for assigning weights. Weights can be assigned in a number of ways depending on the situation and the problem domain.

The next measure of central tendency that we will discuss is the median.

Let us understand this concept with the help of an example.

Let us return to the problem of the 'average' number of floors in the buildings at the centre of a city. We saw that the arithmetic mean was distorted towards the few extremely high values in this series and became unrepresentative.

We could more appropriately and easily employ the median as the 'average' in these circumstances.

MEDIAN:

The median is the middle value of the series when the variable values are placed in order of magnitude.

MEDIAN:

The median is defined as a value which divides a set of data into two halves, one half comprising of observations greater than and the other half smaller than it. More precisely, the median is a value at or below which 50% of the data lie.

The median value can be ascertained by inspection in many series. For instance, in this very example, the data that we obtained was:

EXAMPLE-1:

The average number of floors in the buildings at the centre of a city:

5, 4, 3, 4, 5, 4, 3, 4, 5, 20, 5, 6, 32, 8, 27

Arranging these values in ascending order, we obtain

3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 6, 8, 20, 27, 32

Picking up the middle value, we obtain the median

equal to 5.

Interpretation:

The median number of floors is 5. Out of those 15 buildings, 7 have upto 5 floors and 7 have 5 floors or more. We noticed earlier that the arithmetic mean was distorted toward the few extremely high values in the series and hence became unrepresentative. The median = 5 is much more representative of this series.

EXAMPLE-2:

Height of buildings (number of floors)	
3	
3	
4	
4	**7 lower**
4	
5	
5	
5 = median height	
5	
5	
6	
8	**7 higher**
20	
27	
32	

EXAMPLE-3:

Retail price of motor-car (£) (several makes and sizes)	
415	
480	**4 above**
525	
608	
719 = median price	
1,090	
2,059	
4,000	**4 above**
6,000	

A slight complication arises when there are even numbers of observations in the series, for now there are two middle values.

The expedient of taking the arithmetic mean of the two is adopted as explained below:

EXAMPLE-4

Number of passengers travelling on a bus at six Different times during the day
4
9
14
18 = **median value**
23
47
Median = $\frac{14+18}{2}$ = **16 passengers**

Example -5:

The number of passengers traveling on a bus at six different times during a day are as follows:
5, 14, 47, 34, 18, 23
Find the median.
Solution:
Arranging the values in ascending order, we obtain
5, 14, 18, 23, 34, 47
As before, a slight complication has arisen because of the fact that there are even numbers of observations in the series and, as such, there are two middle values. As before, we take the arithmetic mean of the two middle values.
Hence we obtain:

Median:
$$\tilde{X} = \frac{18+23}{2} = 20.5 \text{ passengers}$$

A very important point to be noted here is that we must arrange the data in ascending order before searching for the two middle values. All the above examples pertained to raw data. Let us now consider the case of grouped data.
We begin by discussing the case of discrete data grouped into a frequency table.
As stated earlier, a discrete frequency distribution is no more than a concise representation of a simple series pertaining to a discrete variable, so that the same approach as the one discussed just now would seem relevant.

EXAMPLE OF A DISCRETE FREQUENCY DISTRIBUTION
Comprehensive School:

Number of pupils per class	Number of Classes
23	1
24	0
25	1
26	3
27	6
28	9
29	8
30	10
31	7
	45

In order to locate the middle value, the best thing is to first of all construct a column of cumulative frequencies:

Comprehensive School

Number of pupils per class X	Number of Classes f	Cumulative Frequency cf
23	1	1
24	0	1
25	1	2
26	3	5
27	6	11
28	9	20
29	8	28
30	10	38
31	7	45
	45	

In this school, there are 45 classes in all, so that we require as the median that class-size below which there are 22 classes and above which also there are 22 classes.

In other words, we must find the 23rd class in an ordered list. We could simply count down noticing that there is 1 class of 23 children, 2 classes with up to 25 children, 5 classes with up to 26 children. Proceeding in this manner, we find that 20 classes contain up to 28 children whereas 28 classes contain up to 29 children. This means that the 23rd class --- the one that we are looking for --- is the one which contains exactly 29 children.

Comprehensive School:

Number of pupils per class X	Number of Classes f	Cumulative Frequency cf
23	1	1
24	0	1
25	1	2
26	3	5
27	6	11
28	9	20
29	8	28
30	10	38
31	7	45
	45	

Median number of pupils per class:

$$\tilde{X} = 29$$

This means that 29 is the middle size of the class. In other words, 22 classes are such which contain 29 or less than 29 children, and 22 classes are such which contain 29 or more than 29 children.

LECTURE # 25

Median, Empirical Relation, Quartiles

- Median in case of a frequency distribution of a continuous variable
- Median in case of an open-ended frequency distribution
- Empirical relation between the mean, median and the mode
- Quantiles (quartiles, deciles & percentiles)
- Graphic location of quantiles.

Median in Case of a Frequency Distribution of a Continuous Variable:

In case of a frequency distribution, the median is given by the formula

$$\tilde{X} = l + \frac{h}{f}\left(\frac{n}{2} - c\right)$$

Where

l =lower class boundary of the median class (i.e. that class for which the cumulative frequency is just in excess of n/2).

h=class interval size of the median class

f =frequency of the median class

n=Σf (the total number of observations)

c =cumulative frequency of the class preceding the median class

Note:

This formula is based on the assumption that the observations in each class are evenly distributed between the two class limits.

Example:

Going back to the example of the EPA mileage ratings, we have

Mileage Rating	No. of Cars	Class Boundaries	Cumulative Frequency
30.0 – 32.9	**2**	**29.95 – 32.95**	**2**
33.0 – 35.9	**4**	**32.95 – 35.95**	**6**
36.0 – 38.9	**14**	**35.95 – 38.95**	**20**
39.0 – 41.9	**8**	**38.95 – 41.95**	**28**
42.0 – 44.9	**2**	**41.95 – 44.95**	**30**

In this example, n = 30 and n/2 = 15.

Thus the third class is the median class. The median lies somewhere between 35.95 and 38.95. Applying the above formula, we obtain

$$\begin{aligned}\tilde{X} &= 35.95 + \frac{3}{14}(15 - 6)\\ &= 35.95 + 1.93\\ &= 37.88\\ &\simeq 37.9\end{aligned}$$

Interpretation:

This result implies that half of the cars have mileage less than or up to 37.88 miles per gallon whereas the other half of the cars has mileage greater than 37.88 miles per gallon. As discussed earlier, the median is preferable to the arithmetic mean when there are a few very high or low figures in a series.

It is also exceedingly valuable when one encounters a frequency distribution having open-ended class intervals.
The concept of open-ended frequency distribution can be understood with the help of the following example.

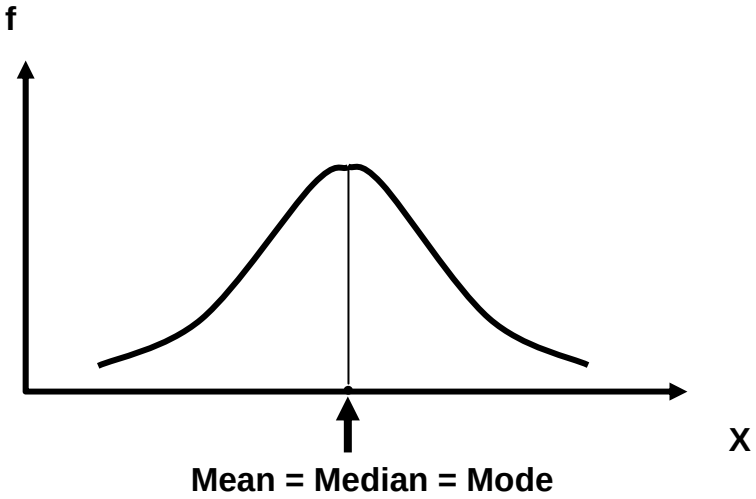
Example:

WAGES OF WORKERS IN A FACTORY	
Monthly Income (in Rupees)	**No. of Workers**
Less than 2000/-	**100**
2000/- to 2999/-	**300**
3000/- to 3999/-	**500**
4000/- to 4999/-	**250**
5000/- and above	**50**
Total	**1200**

In this example, both the first class and the last class are open-ended classes. This is so because of the fact that we do not have exact figures to begin the first class or to end the last class. The advantage of computing the median in the case of an open-ended frequency distribution is that, except in the unlikely event of the median falling within an open-ended group occurring in the beginning of our frequency distribution, there is no need to estimate the upper or lower boundary. This is so because of the fact that, if the median is falling in an intermediate class, then, obviously, the first class is not being involved in its computation.The next concept that we will discuss is the empirical relation between the mean, median and the mode. This is a concept which is not based on a rigid mathematical formula; rather, it is based on observation. In fact, the word 'empirical' implies 'based on observation'.

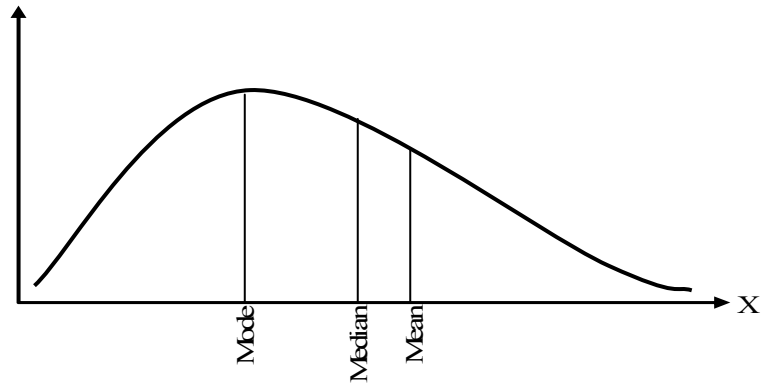
This concept relates to the relative positions of the mean, median and the mode in case of a hump-shaped distribution. In a single-peaked frequency distribution, the values of the mean, median and mode coincide if the frequency distribution is absolutely symmetrical.

THE SYMMETRIC CURVE



But in the case of a skewed distribution, the mean, median and mode do not all lie on the same point. They are pulled apart from each other, and the empirical relation explains the way in which this happens. Experience tells us that in a unimodal curve of moderate skewness, the median is usually sandwiched between the mean and the mode.

The second point is that, in the case of many real-life data-sets, it has been observed that the distance between the mode and the median is approximately double of the distance between the median and the mean, as shown below:



This diagrammatic picture is equivalent to the following algebraic expression:
Median - Mode 2 (Mean - Median) ---- (1)
The above-mentioned point can also be expressed in the following way:

Mean – Mode = 3 (Mean – Median) ---- (2)

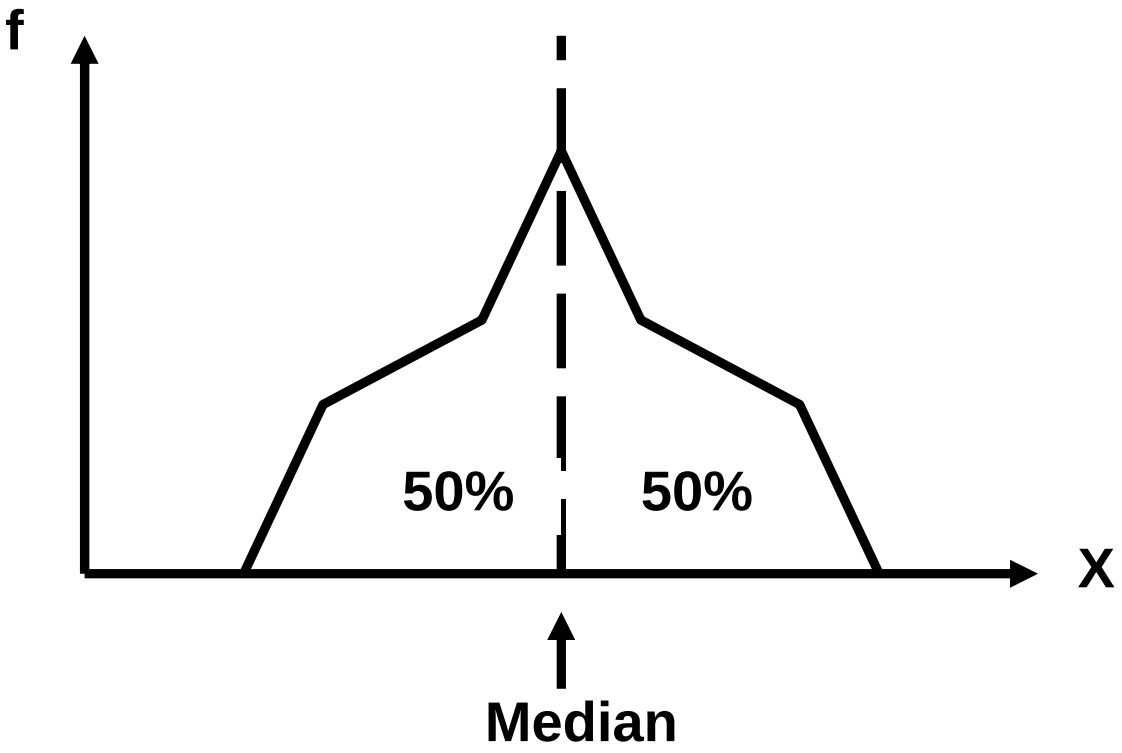
Equation (1) as well as equation (2) yields the approximate relation given below:

EMPIRICAL RELATION BETWEEN
THE MEAN, MEDIAN AND THE MODE :

Mode = 3 Median – 2 Mean

An exactly similar situation holds in case of a moderately negatively skewed distribution.
An important point to note is that this empirical relation does not hold in case of a J-shaped or an extremely skewed distribution.

Let us now extend the concept of partitioning of the frequency distribution by taking up the concept of quantiles (i.e. quartiles, deciles and percentiles).
We have already seen that the median divides the area under the frequency polygon into two equal halves:



A further split to produce quarters, tenths or hundredths of the total area under the frequency polygon is equally possible, and may be extremely useful for analysis. (We are often interested in the highest 10% of some group of values or the middle 50% another.)

QUARTILES

The quartiles, together with the median, achieve the division of the total area into four equal parts. The first, second and third quartiles are given by the formulae:

First quartile:

$$Q_1 = l + \frac{h}{f}\left(\frac{n}{4} - c\right)$$

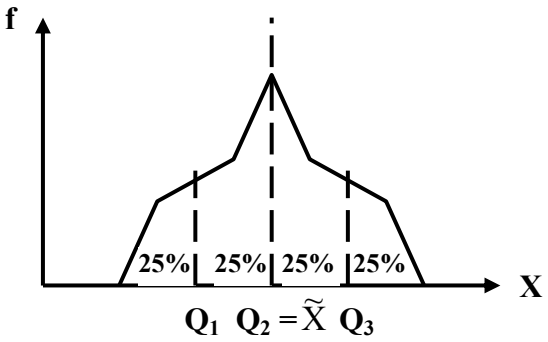
Second quartile (i.e. median):

$$Q_2 = l + \frac{h}{f}\left(\frac{2n}{4} - c\right) = l + \frac{h}{f}(n/2 - c)$$

Third quartile:

$$Q_3 = l + \frac{h}{f}\left(\frac{3n}{4} - c\right)$$

It is clear from the formula of the second quartile that the second quartile is the same as the median.



DECILES & PERCENTILES

The deciles and the percentiles given the division of the total area into 10 and 100 equal parts respectively.

The formula for the first decile is

$$D_1 = l + \frac{h}{f}\left(\frac{n}{10} - c\right)$$

The formulae for the subsequent deciles are

$$D_2 = l + \frac{h}{f}\left(\frac{2n}{10} - c\right)$$

$$D_3 = l + \frac{h}{f}\left(\frac{3n}{10} - c\right)$$

and so on.

It is easily seen that the 5th decile is the same quantity as the median.

The formula for the first percentile is

$$P_1 = l + \frac{h}{f}\left(\frac{n}{100} - c\right)$$

The formulae for the subsequent percentiles are

$$P_2 = l + \frac{h}{f}\left(\frac{2n}{100} - c\right)$$

$$P_3 = l + \frac{h}{f}\left(\frac{3n}{100} - c\right)$$

and so on.

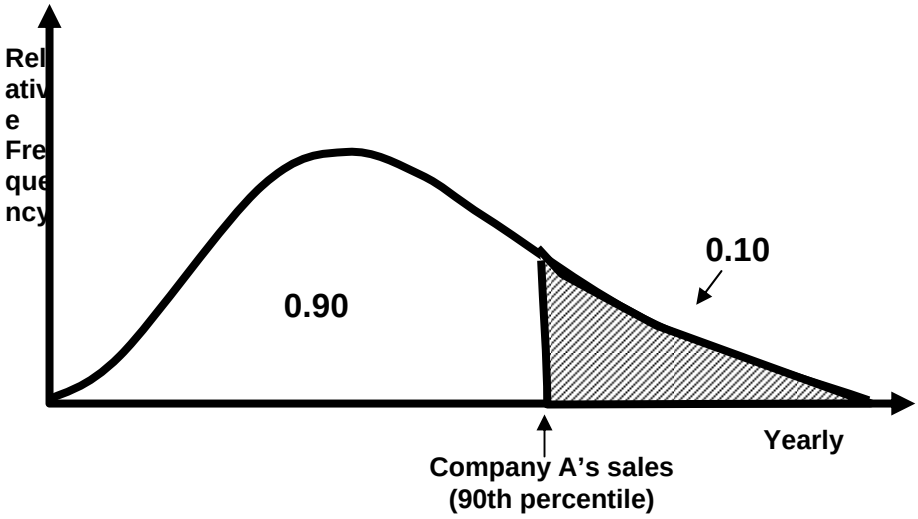
Again, it is easily seen that the 50th percentile is the same as the median, the 25th percentile is the same as the 1st quartile, the 75th percentile is the same as the 3rd quartile, the 40th percentile is the same as the 4th decile, and so on.

All these measures i.e. the median, quartiles, deciles and percentiles are collectively called quantiles. The question is, "What is the significance of this concept of partitioning? *Why* is it that we wish to divide our frequency distribution into two, four, ten or hundred parts?"

The answer to the above questions is: In certain situations, we may be interested in describing the *relative* quantitative location of a particular measurement within a data set. Quantiles provide us with

an easy way of achieving this. Out of these various quantiles, one of the most frequently used is percentile ranking.
Let us understand this point with the help of an example.

EXAMPLE:
If oil company 'A' reports that its yearly sales are at the 90th percentile of all companies in the industry, the implication is that 90% of all oil companies have yearly sales *less* than company A's, and only 10% have yearly sales exceeding company A's:
This is demonstrated in the following figure:



It is evident from the above example that the concept of percentile ranking is quite a useful concept, but it should be kept in mind that percentile rankings are of practical value only for large data sets.
It is evident from the above example that the concept of percentile ranking is quite a useful concept, but it should be kept in mind that percentile rankings are of practical value only for large data sets.
The next concept that we will discuss is the graphic location of quantiles.
Let us go back to the example of the EPA mileage ratings of 30 cars that was discussed in an earlier lecture. The statement of the example was:
EXAMPLE:
Suppose that the Environmental Protection Agency of a developed country performs extensive tests on all new car models in order to determine their mileage rating.
Suppose that the following 30 measurements are obtained by conducting such tests on a particular new car model.

EPA MILEAGE RATINGS ON 30 CARS (MILES PER GALLON)		
36.3	42.1	44.9
30.1	37.5	32.9
40.5	40.0	40.2
36.2	35.6	35.9
38.5	38.8	38.6
36.3	38.4	40.5
41.0	39.0	37.0
37.0	36.7	37.1
37.1	34.8	33.9
39.9	38.1	39.8

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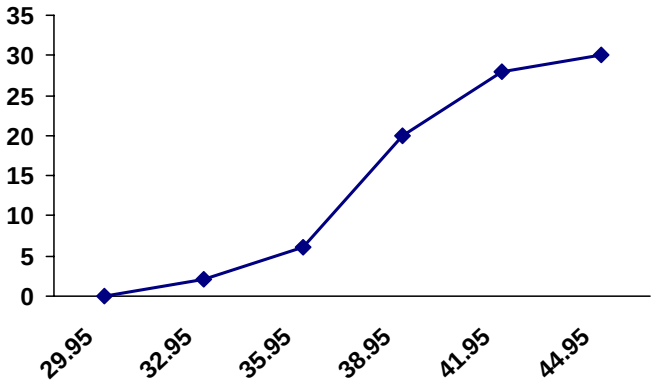
When the above data was converted to a frequency distribution, we obtained:

Class Limit	Frequency
30.0 – 32.9	**2**
33.0 – 35.9	**4**
36.0 – 38.9	**14**
39.0 – 41.9	**8**
42.0 – 44.9	**2**
	30

Also, we considered the graphical representation of this distribution.
The cumulative frequency polygon of this distribution came out to be as shown in the following figure:

Cumulative Frequency Polygon or OGIVE



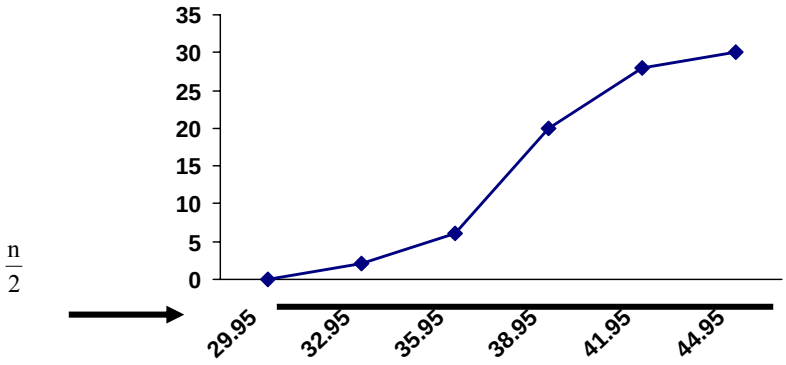
This ogive enables us to find the median and any other quantile that we may be interested in very conveniently. And this process is known as the graphic location of quantiles.
Let us begin with the graphical location of the median:

Because of the fact that the median is that value before which half of the data lies, the first step is to divide the total number of observations n by 2.
In this example:

$$\frac{n}{2} = \frac{30}{2} = 15$$

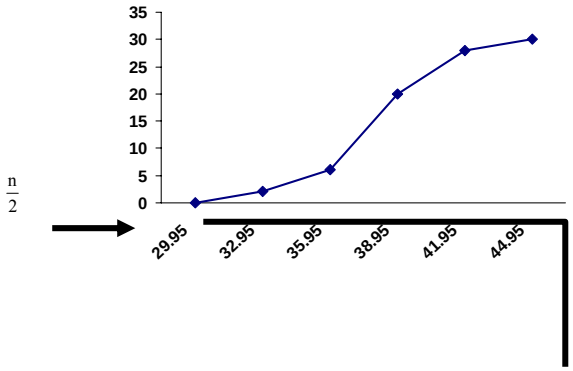
The next step is to locate this number 15 on the y-axis of the cumulative frequency polygon.

Cumulative Frequency Polygon or OGIVE



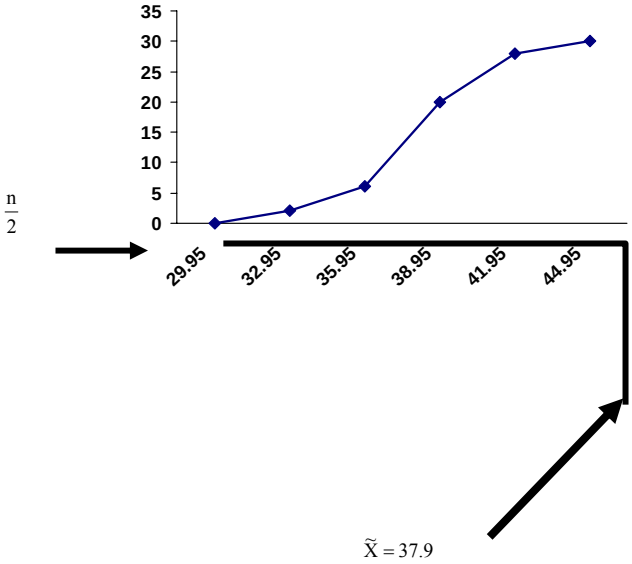
Lastly, we drop a vertical line from the cumulative frequency polygon down to the x-axis.

Cumulative Frequency Polygon or OGIVE



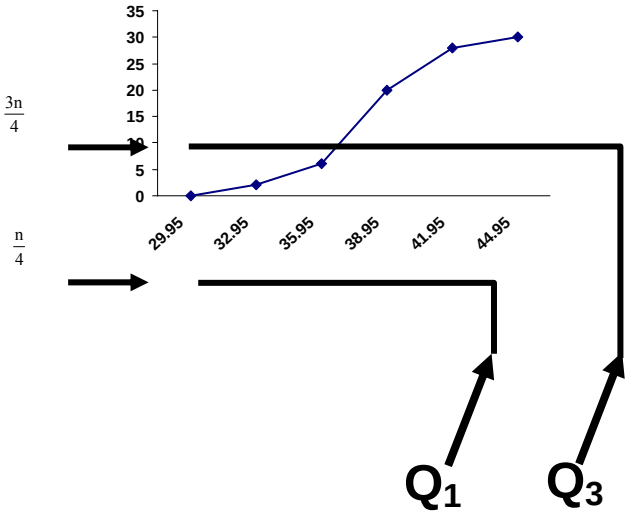
Now, if we read the x-value where our perpendicular touches the x-axis, students, we find that this value is approximately the same as what we obtained from our formula.

Cumulative Frequency Polygon or OGIVE



It is evident from the above example that the cumulative frequency polygon is a very useful device to find the value of the median very quickly.In a similar way, we can locate the quartiles, deciles and percentiles.To obtain the first quartile, the horizontal line will be drawn against the value n/4, and for the third quartile, the horizontal line will be drawn against the value 3n/4.

Cumulative Frequency Polygon or OGIVE



For the deciles, the horizontal lines will be against the values n/10, 2n/10, 3n/10, and so on. And for the percentiles, the horizontal lines will be against the values n/100, 2n/100, 3n/100, and so on.

The graphic location of the quartiles as well as of a few deciles and percentiles for the data-set of the EPA mileage ratings may be taken up as an exercise:

This brings us to the end of our discussion regarding quantiles which are sometimes also known as fractiles --- this terminology because of the fact that they divide the frequency distribution into various parts or fractions.

LECTURE # 26

Geometric mean, Harmonic mean & relationship between them

- Geometric mean
- Harmonic mean
- Relation between the arithmetic, geometric and harmonic means
- Some other measures of central tendency

GEOMETRIC MEAN:

The geometric mean, G, of a set of n positive values X1, X2,…,Xn is defined as the positive nth root of their product.

$$G = \sqrt[n]{X_1 X_2 ... X_n}$$

(Where $X_i > 0$)

When n is large, the computation of the geometric mean becomes laborious as we have to extract the nth root of the product of all the values.

The arithmetic is simplified by the use of logarithms.

Taking logarithms to the base 10, we get

$$\log G = \frac{1}{n}\left[\log X_1 + \log X_2 + ... + \log X_n\right]$$

Hence

$$= \frac{\sum \log\ X}{n}$$

$$G = \text{anti}\log\left[\frac{\sum \log\ X}{n}\right]$$

Example:

Find the geometric mean of numbers:

45, 32, 37, 46, 39,
36, 41, 48, 36.

Solution:

We need to compute the numerical value of

$$= \sqrt[9]{45\times 32\times 37\times 46\times 39\times 36\times 41\times 48\times 36}$$

But, obviously, it is a bit cumbersome to find the ninth root of a quantity. So we make use of logarithms, as shown below:

X	log X
45	1.6532
32	1.5052
37	1.5682
46	1.6628
39	1.5911
36	1.5563
41	1.6128
48	1.6812
36	1.5563
	14.3870

$$\log G = \frac{\sum \log X}{n}$$

$$= \frac{14.3870}{9} = 1.5986$$

$$\text{Hence } G = \text{anti}\log\ 1.5986$$

$$= 39.68$$

The above example pertained to the computation of the geometric mean in case of raw data.
Next, we consider the computation of the geometric mean in the case of grouped data.

GEOMETRICMEAN
FOR GROUPED DATA:

In case of a frequency distribution having k classes with midpoints X1, X2, …,Xk and the corresponding frequencies f1, f2, …, fk (such that ∑fi = n), the geometric mean is given by

$$G = \sqrt[n]{X_1^{f_1} X_2^{f_2} \ldots X_k^{f_k}}$$

Each value of X thus has to be multiplied by itself f times, and the whole procedure becomes quite a formidable task!
In terms of logarithms, the formula becomes

$$\log G = \frac{1}{n}\left[f_1 \log X_1 + f_2 \log X_2 + \ldots + f_k \log X_k\right]$$

$$= \frac{\sum f \log X}{n}$$

Hence

$$G = antilog\left[\frac{\sum f \log X}{n}\right]$$

Obviously, the above formula is much easier to handle.
Let us now apply it to an example.
Going back to the example of the EPA mileage ratings, we have:

Mileage Rating	No. of Cars	Class-mark (midpoint) X	log X	f log X
30.0 - 32.9	2	31.45	1.4976	2.9952
33.0 - 35.9	4	34.45	1.5372	6.1488
36.0 - 38.9	14	37.45	1.5735	22.0290
39.0 - 41.9	8	40.45	1.6069	12.8552
42.0 - 44.9	2	43.45	1.6380	3.2760
	30			47.3042

$$G = \text{antilog}\ \frac{47.3042}{30}$$

= antilog 1.5768 = 37.74
This means that, if we use the geometric mean to measures the central tendency of this data set, then the central value of the mileage of those 30 cars comes out to be 37.74 miles per gallon.
The question is, “When should we use the geometric mean?”

The answer to this question is that when relative changes in some variable quantity are averaged, we prefer the geometric mean.

EXAMPLE:

Suppose it is discovered that a firm's turnover has increased during 4 years by the following amounts:

Year	Turnover	Percentage Compared With Year Earlier
1958	£ 2,000	–
1959	£ 2,500	125
1960	£ 5,000	200
1961	£ 7,500	150
1962	£ 10,500	140

The yearly increase is shown in a percentage form in the right-hand column i.e. the turnover of 1959 is 125 percent of the turnover of 1958, the turnover of 1960 is 200 percent of the turnover of 1959, and so on. The firm's owner may be interested in knowing his average rate of turnover growth.

If the arithmetic mean is adopted he finds his answer to be:

Arithmetic Mean:

$$\frac{125+200+150+140}{4}$$

$$=153.75$$

i.e. we are concluding that the turnover for any year is 153.75% of the turnover for the previous year. In other words, the turnover in each of the years considered appears to be 53.75 per cent higher than in the previous year.

If *this* percentage is used to calculate the turnover from 1958 to 1962 inclusive, we obtain:

153.75% of £ 2,000 = £ 3,075
153.75% of £ 3,075 = £ 4,728
153.75% of £ 4,728 = £ 7,269
153.75% of £ 7,269 = £ 11,176

Whereas the actual turnover figures were

Year	Turnover
1958	£ 2,000
1959	£ 2,500
1960	£ 5,000
1961	£ 7,500
1962	£ 10,500

It seems that both the individual figures and, more important, the total at the end of the period, are incorrect. Using the arithmetic mean has exaggerated the 'average' annual rate of increase in the turnover of this firm. Obviously, we would like to rectify this false impression. The geometric mean enables us to do so:

Geometric mean of the turnover figures:

$$\sqrt[4]{(125\times 200\times 150\times 140)}$$
$$= \sqrt[4]{525000000}$$
$$= 151.37\%$$

Now, if we utilize this particular value to obtain the individual turnover figures, we find that:
151.37% of £2,000 = £3,027
151.37% of £3,027 = £4,583
151.37% of £4,583 = £6,937
151.37% of £6,937 = £10,500
So that the turnover figure of 1962 is exactly the same as what we had in the original data.

Interpretation:

If the turnover of this company were to increase annually at a constant rate, then the annual increase would have been 51.37 percent.(On the average, each year's turnover is 51.37% higher than that in the previous year.) The above example clearly indicates the significance of the geometric mean in a situation when relative changes in a variable quantity are to be averaged.

But we should bear in mind that such situations are not encountered too often, and that the occasion to calculate the geometric mean arises less frequently than the arithmetic mean.(The most frequently used measure of central tendency is the arithmetic mean.)

The next measure of central tendency that we will discuss is the harmonic mean.

HARMONIC MEAN;

The harmonic mean is defined as the reciprocal of the arithmetic mean of the reciprocals of the values. **HARMONIC MEAN**

In case of raw data:

$$H.M. = \frac{n}{\sum\left(\frac{1}{X}\right)}$$

In case of grouped data (data grouped into a frequency distribution):

$$H.M. = \frac{n}{\sum f\left(\frac{1}{X}\right)}$$

(where X represents the midpoints of the various classes).

EXAMPLE:

Suppose a car travels 100 miles with 10 stops, each stop after an interval of 10 miles. Suppose that the speeds at which the car travels these 10 intervals are 30, 35, 40, 40, 45, 40, 50, 55, 55 and 30 miles per hours respectively.

What is the average speed with which the car traveled the total distance of 100 miles?

If we find the arithmetic mean of the 10 speeds, we obtain:

Arithmetic mean of the 10 speeds:

$$\frac{30+35+....+30}{10}$$
$$= \frac{420}{10} = 42 \text{ miles per hour.}$$

But, if we study the problem carefully, we find that the above answer is incorrect.

By definition, the average speed is the speed with which the car would have traveled the 100 mile distance if it had maintained a constant speed throughout the 10 intervals of 10 miles each.

$$\textbf{Average speed} = \frac{\textbf{Total distance travelled}}{\textbf{Total time taken}}$$

Now, *total distance traveled* = 100 miles.
Total time taken will be computed as shown below:

Interval	Distance	Speed = $\frac{\text{Distance}}{\text{Time}}$	Time = $\frac{\text{Distance}}{\text{Speed}}$
1	**10 miles**	**30 mph**	**10/30 = 0.3333 hrs**
2	**10 miles**	**35 mph**	**10/35 = 0.2857 hrs**
3	**10 miles**	**40 mph**	**10/40 = 0.2500 hrs**
4	**10 miles**	**40 mph**	**10/40 = 0.2500 hrs**
5	**10 miles**	**45 mph**	**10/45 = 0.2222 hrs**
6	**10 miles**	**40 mph**	**10/40 = 0.2500 hrs**
7	**10 miles**	**50 mph**	**10/50 = 0.2000 hrs**
8	**10 miles**	**55 mph**	**10/55 = 0.1818 hrs**
9	**10 miles**	**55 mph**	**10/55 = 0.1818 hrs**
10	**10 miles**	**30 mph**	**10/30 = 0.333 hrs**
Total = 100 miles		**Total Time = 2.4881 hrs**	

Hence
Average speed = $\frac{100}{2.4881} = 40.2\ mph$

which is not the same as 42 miles per hour.
Let us now try the harmonic mean to find the average speed of the car.

$$H.M. = \frac{n}{\sum\left(\frac{1}{X}\right)}$$

where n is the no. of term.
We have:

X	1/X
30	1/30 = 0.0333
35	1/35 = 0.0286
40	1/40 = 0.0250
40	1/40 = 0.0250
45	1/45 = 0.0222
40	1/40 = 0.0250
50	1/50 = 0.0200
55	1/55 = 0.0182
55	1/55 = 0.0182
30	1/30 = 0.0333
	$\sum \frac{1}{X} = 0.2488$

$$\text{H.M.} = \frac{n}{\sum \frac{1}{X}}$$

$$= \frac{10}{0.2488}$$

= 40.2 mph

Hence it is clear that the harmonic mean gives the totally correct result**.**

The key question is, "When should we compute the harmonic mean of a data set?"
The answer to this question will be easy to understand if we consider the following rules:

RULES

1. When values are given as x per y where x is constant and y is variable, the Harmonic Mean is the appropriate average to use.
2. When values are given as x per y where y is constant and x is variable, the Arithmetic Mean is the appropriate average to use.
3. When relative changes in some variable quantity are to be averaged, the geometric mean is the appropriate average to use.

We have already discussed the geometric and the harmonic means.
Let us now try to understand Rule No. 1 with the help of an example:

EXAMPLE:

If 10 students have obtained the following marks (in a test) out of 20:
13, 11, 9, 9, 6,
5, 19, 17, 12, 9
Then the average marks (by the formula of the arithmetic mean) are:

$$\frac{13+11+9+9+6+5+19+17+12+9}{10}$$

$$= \frac{110}{10} = 11$$

This is equivalent to

$$\frac{\frac{13}{20}+\frac{11}{20}+\frac{9}{20}+\frac{9}{20}+\frac{6}{20}+\frac{5}{20}+\frac{19}{20}+\frac{17}{20}+\frac{12}{20}+\frac{9}{20}}{10}$$

$$= \frac{\frac{110}{20}}{10} = \frac{110}{10 \times 20} = \frac{11}{20}$$

(i.e. the average marks of this group of students are 11 out of 20).
In the above example, the point to be noted was that all the marks were expressible as x per y where the denominator y was constant i.e. equal to 20, and hence, it was appropriate to compute the arithmetic mean.
Let us now consider a mathematical relationship exists between these three measures of central tendency.

RELATION BETWEEN ARITHMETIC, GEOMETRIC AND HARMONIC MEANS:

Arithmetic Mean ≥ Geometric Mean ≥Harmonic Mean

We have considered the five most well-known measures of central tendency i.e. arithmetic mean, median, mode, geometric mean and harmonic mean.
It is interesting to note that there are some other measures of central tendency as well.
Two of these are the mid range, and the mid quartile range.

Let us consider these one by one:

MID-RANGE:

If there are n observations with x0 and xm as their smallest and largest observations respectively, then their mid-range is defined as

$$\text{mid} - \text{range} = \frac{x_0 + x_m}{2}$$

It is obvious that if we add the smallest value with the largest, and divide by 2, we will get a value which is more or less in the middle of the data-set.

MID-QUARTILE RANGE:

If x1, x2… xn are n observations with Q1 and Q3 as their first and third quartiles respectively, then their mid-quartile range is defined as

$$\text{mid} - \text{quartile range} = \frac{Q_1 + Q_3}{2}$$

Similar to the case of the mid-range, if we take the arithmetic mean of the upper and lower quartiles, we will obtain a value that is somewhere in the middle of the data-set.
The mid-quartile range is also known as the mid-hinge.

Let us now revise briefly the core concept of central tendency:

Masses of data are usually expressed in the form of frequency tables so that it becomes easy to comprehend the data.
Usually, a statistician would like to go a step ahead and to compute a number that will represent the data in some definite way.
Any such single number that represents a whole set of data is called **'Average'**.
Technically speaking, there are many kinds of averages (i.e. there are several ways to compute them).
These quantities that represent the data-set are called "measures of central tendency".

LECTURE # 27

Measures of Dispersion

- Concept of dispersion
- Absolute and relative measures of dispersion
- Range
- Coefficient of dispersion
- Quartile deviation
- Coefficient of quartile deviation

Let us begin the concept of DISPERSION.

Just as variable series differ with respect to their location on the horizontal axis (having different 'average' values); similarly, they differ in terms of the amount of variability which they exhibit.

Let us understand this point with the help of an example:

EXAMPLE:

In a technical college, it may well be the case that the ages of a group of first-year students are quite consistent, e.g. 17, 18, 18, 19, 18, 19, 19, 18, 17, 18 and 18 years.

A class of evening students undertaking a course of study in their spare time may show just the opposite situation, e.g. 35, 23, 19, 48, 32, 24, 29, 37, 58, 18, 21 and 30.

It is very clear from this example that the variation that exists between the various values of a data-set is of substantial importance. We obviously need to be aware of the amount of variability present in a data-set if we are to come to useful conclusions about the situation under review. This is perhaps best seen from studying the two frequency distributions given below:

EXAMPLE:

The sizes of the classes in two comprehensive schools in different areas are as follows:

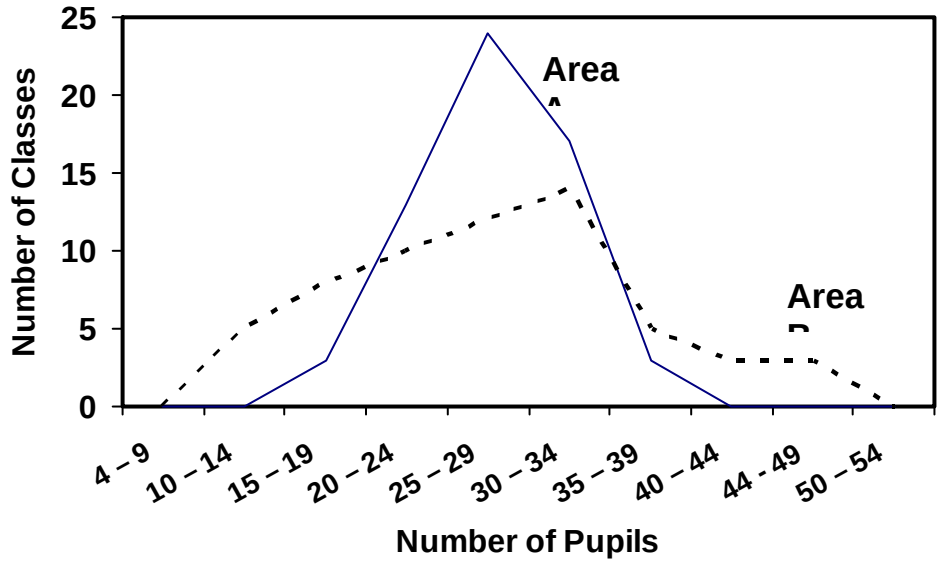
Number of Pupils	Number of Classes	
	Area A	Area B
10 – 14	0	5
15 – 19	3	8
20 – 24	13	10
25 – 29	24	12
30 – 34	17	14
35 – 39	3	5
40 – 44	0	3
45 - 49	0	3
	60	60

If the arithmetic mean size of class is calculated, we discover that the answer is identical: 27.33 pupils in both areas.

Average class-size of each school

$$\overline{X} = 27.33$$

Even though these two distributions share a common average, it can readily be seen that they are entirely DIFFERENT.
And the graphs of the two distributions (given below) clearly indicate this fact.



The question which must be posed and answered is 'In what way can these two situations be distinguished?'

We need a measure of variability or DISPERSION to accompany the relevant measure of position or 'average' used.

The word 'relevant' is important here for we shall find one measure of dispersion which expresses the scatter of values round the arithmetic mean, another the scatter of values round the median, and so forth. Each measure of dispersion is associated with a particular 'average'.

Absolute versus Relative Measures of Dispersion:

There are two types of measurements of dispersion: absolute and relative.

An ***absolute*** measure of dispersion is one that measures the dispersion in terms of the same units or in the square of units, as the units of the data.

For example, if the units of the data are rupees, meters, kilograms, etc., the units of the measures of dispersion will also be rupees, meters, kilograms, etc.

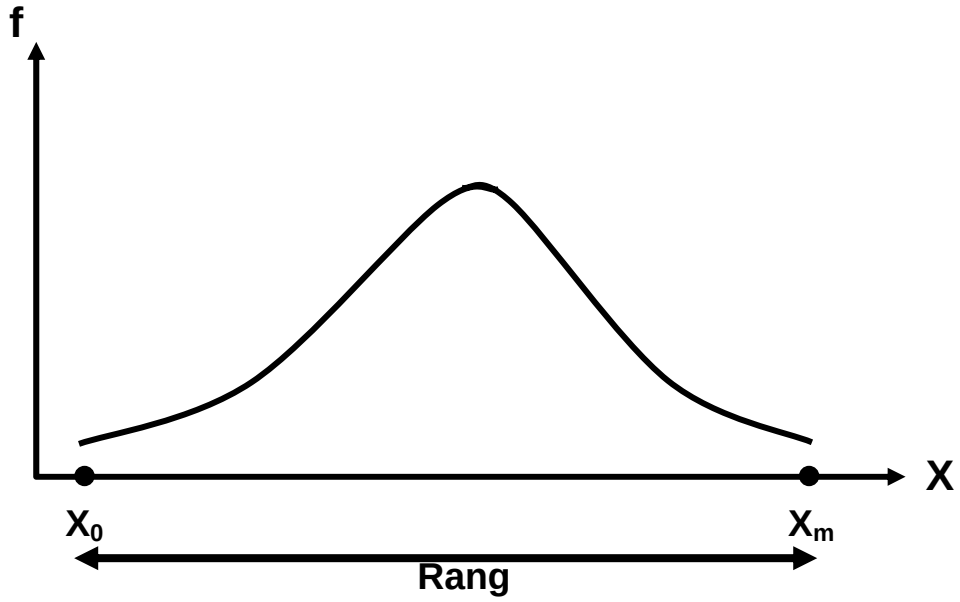
On the other hand, ***relative*** measure of dispersion is one that is expressed in the form of a ratio, co-efficient of percentage and is independent of the units of measurement.

A *relative* measure of dispersion is useful for comparison of data of different nature. A measure of central tendency together with a measure of dispersion gives an adequate description of data. We will be discussing FOUR measures of dispersion i.e. the range, the quartile deviation, the mean deviation, and the standard deviation.

RANGE:

The range is defined as the difference between the two extreme values of a data-set, i.e. R = Xm – X0 where Xm represents the highest value and X0 the lowest.
Evidently, the calculation of the range is a simple question of MENTAL arithmetic.

The simplicity of the concept does not necessarily invalidate it, but in general it gives no idea of the DISTRIBUTION of the observations between the two ends of the series. For this reason it is used principally as a supplementary aid in the description of variable data, in conjunction with other measures of dispersion. When the data are grouped into a frequency distribution, the range is estimated by finding the difference between the upper boundary of the highest class and the lower boundary of the lowest class.
We now consider the graphical representation of the range:



Obviously, the greater the difference between the largest and the smallest values, the greater will be the range. As stated earlier, the range is a simple concept and is easy to compute. However, because of the fact that it is computed from only the two extreme values in a data-set, it has two serious disadvantages.

1. It *ignores* all the INFORMATION available from the intermediate observations.
2. It might give a MISLEADING picture of the spread in the data.

From THIS point of view, it is an unsatisfactory measure of dispersion. However, it is APPROPRIATELY used in statistical quality control charts of manufactured products, daily temperatures, stock prices, etc.

It is interesting to note that the range can also be viewed in the following way:

It is twice of the arithmetic mean of the deviations of the smallest and largest values round the mid-range i.e.

$$\frac{(\text{Midrange} - X_0) + (X_m - \text{Midrange})}{2}$$

$$= \frac{\text{Midrange} - X_0 + X_m - \text{Midrange}}{2}$$

$$= \frac{X_m - X_0}{2}$$

Because of what has been just explained, the range can be regarded as that measure of dispersion which is associated with the mid-range. As such, the range may be employed to indicate dispersion when the mid-range has been adopted as the most appropriate average.

The range is an *absolute* measure of dispersion. Its *relative* measure is known as the CO-EFFICIENT OF DISPERSION, and is defined by the relation given below:

Coefficient of Dispersion:

$$= \frac{\frac{1}{2}(\text{Range})}{\text{Mid} - \text{Range}}$$

$$= \frac{\frac{X_m - X_0}{2}}{\frac{X_m + X_0}{2}} = \frac{X_m - X_0}{X_m + X_0}$$

This is a *pure* (i.e. *dimensionless) number* and is used for the purposes of *COMPARISON*. (This is so because a pure number *can* be compared with another pure number.)

For example, if the coefficient of dispersion for one data-set comes out to be 0.6 whereas the coefficient of dispersion for another data-set comes out to be 0.4, then it is obvious that there is greater amount of dispersion in the first data-set as compared with the second.

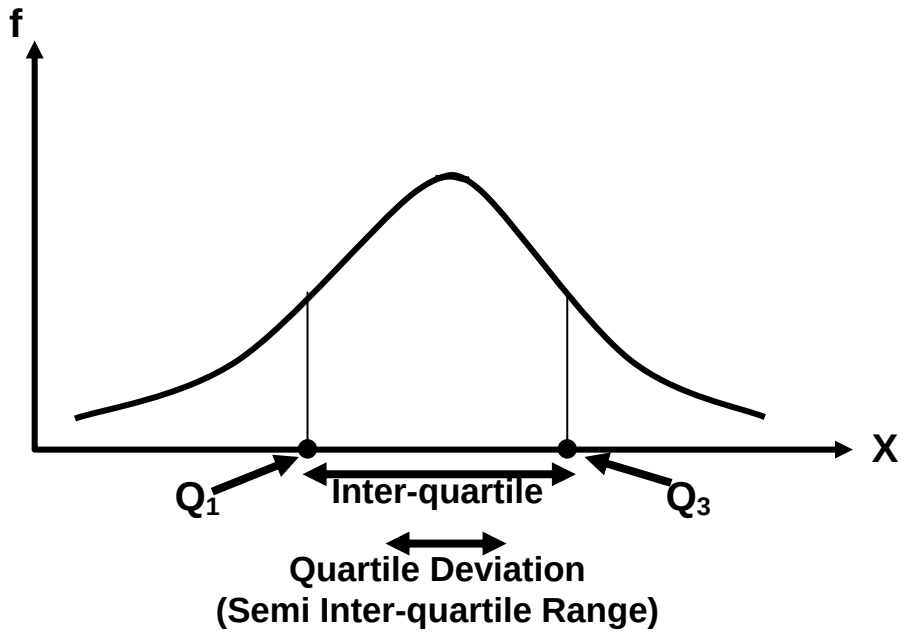
QUARTILE DEVIATION:

The quartile deviation is defined as half of the difference between the third and first quartiles i.e.

$$Q.D. = \frac{Q_3 - Q_1}{2}$$

It is also known as **semi-interquartile range.**

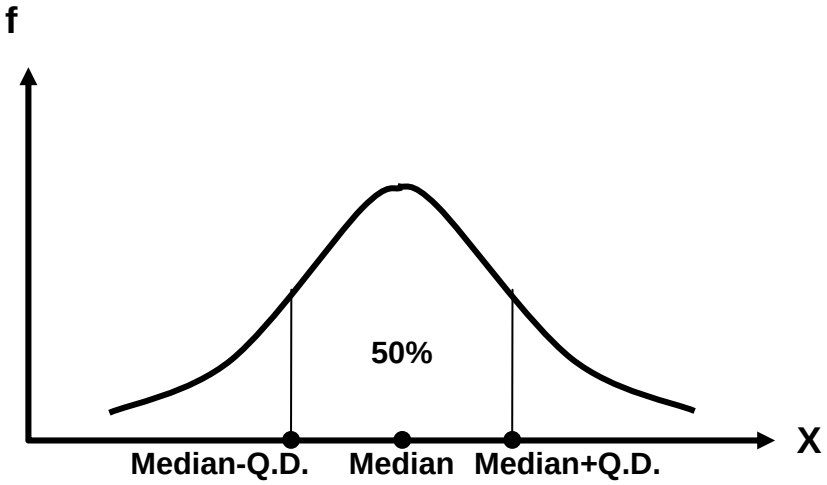
Let us now consider the graphical representation of the quartile deviation:



Although simple to compute, it is NOT an extremely satisfactory measure of dispersion because it takes into account the spread of only two values of the variable round the median, and this gives no idea of the rest of the dispersion within the distribution.

The quartile deviation has an attractive feature that the range "Median + Q.D." contains approximately 50% of the data.

This is illustrated in the figure given below:



Let us now apply the concept of quartile deviation to the following example:

EXAMPLE:

The shareholding structure of two companies is given below:

	Company X	**Company Y**
1st quartile	60 shares	165 shares
Median	185 shares	185 shares
3rd quartile	270 shares	210 shares

The quartile deviation for company X is

$$\frac{270-60}{2}=105 \qquad \text{Shares}$$

For company Y, it is

$$\frac{210-165}{2}=22 \qquad \text{Shares}$$

A comparison of the above two results indicate that there is a considerable concentration of shareholders about the MEDIAN number of shares in company Y, whereas in company X, there does not exist this kind of a concentration around the median. (In company X, there is approximately the SAME numbers of small, medium and large shareholders.)

From the above example, it is obvious that the larger the quartile deviation, the greater is the scatter of values within the series. The quartile deviation is superior to range as it is not affected by extremely large or small observations. It is simple to understand and easy to calculate.

The mean deviation can also be viewed in another way:

It is the arithmetic mean of the deviations of the first and third quartiles round the median i.e.

$$\frac{(M - Q_1) + (Q_3 - M)}{2}$$
$$= \frac{M - Q_1 + Q_3 - M}{2}$$
$$= \frac{Q_3 - Q_1}{2}$$

Because of what has been just explained, the quartile deviation is regarded as that measure of dispersion which is associated with the median. As such, the quartile deviation should always be employed to indicate dispersion when the median has been adopted as the most appropriate average.

The quartile deviation is also an *absolute* measure of dispersion. Its *relative* measure called the CO-EFFICIENT OF QUARTILE DEVIATION or of Semi-Inter-quartile Range, is defined by the relation:

Coefficient of Quartile Deviation:

$$= \frac{Quartile\ Deviation}{Mid - Quartile\ Range}$$
$$= \frac{\frac{Q_3 - Q_1}{2}}{\frac{Q_3 + Q_1}{2}} = \frac{Q_3 - Q_1}{Q_3 + Q_1},$$

The Coefficient of Quartile Deviation is a pure number and is used for *COMPARING* the variation in two or more sets of data.

The next two measures of dispersion to be discussed are the Mean Deviation and the Standard Deviation. In this regard, the first thing to note is that, whereas the range as well as the quartile deviation are two such measures of dispersion which are NOT based on all the values, the mean deviation and the standard deviation are two such measures of dispersion that involve each and every data-value in their computation.

The range measures the dispersion of the data-set around the *mid-range*, whereas the quartile deviation measures the dispersion of the data-set around the *median*.

How are we to decide upon the amount of dispersion round the *arithmetic mean*?

It would seem reasonable to compute the *DISTANCE* of each observed value in the series from the arithmetic mean of the series.

But the problem is that the sum of the deviations of the values from the mean is ZERO! (No matter what the amount of dispersion in a data-set is, this quantity will *always* be zero, and hence it cannot be used to measure the dispersion in the data-set.)

Then, the question arises, 'HOW will we be able to measure the dispersion present in our data-set?'

In an attempt to answer this question, we might look at the numerical differences between the mean and the data values WITHOUT considering whether these are positive or negative. By ignoring the sign of the deviations we will achieve a NON-ZERO sum, and averaging these absolute differences, again, we obtain a non-zero quantity which can be used as a measure of dispersion. (The larger this quantity, the greater is the dispersion in the data-set).

This quantity is known as the **MEAN DEVIATION.**

Let us denote these absolute differences
by 'modulus of d'
or 'mod d'. Then, the mean deviation is given by

MEAN DEVIATION:

$$\text{M.D.} = \frac{\sum |d|}{n}$$

As the absolute deviations of the observations from their mean are being averaged, therefore the complete name of this measure is Mean Absolute Deviation --- but generally, it is simply called "Mean Deviation". In the next lecture, this concept will be discussed in detail. (The case of raw data as well as the case of grouped data will be considered.)Next, we will discuss the most important and the most widely used measure of dispersion i.e. the Standard Deviation.

LECTURE # 28

Mean Deviation, Standard Deviation and Variance & Coefficient of variation

- Mean Deviation
- Standard Deviation and Variance
- Coefficient of variation

First, we will discuss it for the case of raw data, and then we will go on to the case of a frequency distribution. The first thing to note is that, whereas the range as well as the quartile deviation are two such measures of dispersion which are NOT based on all the values, the mean deviation and the standard deviation are two such measures of dispersion that involve each and every data-value in their computation.

You must have noted that the range was measuring the dispersion of the data-set around the mid-range, whereas the quartile deviation was measuring the dispersion of the data-set around the median.

How are we to decide upon the amount of dispersion round the arithmetic mean? It would seem reasonable to compute the DISTANCE of each observed value in the series from the arithmetic mean of the series.

Let us do this for a simple data-set shown below:

The Number of Fatalities in Motorway Accidents in one Week:

Day	Number of fatalities X
Sunday	4
Monday	6
Tuesday	2
Wednesday	0
Thursday	3
Friday	5
Saturday	8
Total	28

Let us do this for a simple data-set shown below:

The Number of Fatalities in Motorway Accidents in one Week:

Day	Number of fatalities X
Sunday	4
Monday	6
Tuesday	2
Wednesday	0
Thursday	3
Friday	5
Saturday	8
Total	28

The arithmetic mean number of fatalities per day is

$$\overline{X} = \frac{\sum X}{n} = \frac{28}{7} = 4$$

In order to determine the distances of the data-values from the mean, we subtract our value of the arithmetic mean from each daily figure, and this gives us the deviations that occur in the third column of the table below:

Day	Number of fatalities X	$X - \overline{X}$
Sunday	**4**	**0**
Monday	**6**	**+ 2**
Tuesday	**2**	**– 2**
Wednesday	**0**	**– 4**
Thursday	**3**	**– 1**
Friday	**5**	**+ 1**
Saturday	**8**	**+ 4**
TOTAL	**28**	**0**

The deviations are negative when the daily figure is less than the mean (4 accidents) and positive when the figure is higher than the mean.
It does seem, however, that our efforts for computing the dispersion of this data set have been in vain, for we find that the total amount of dispersion obtained by summing the $(x - \overline{x})$ column comes out to be zero! In fact, this should be no surprise, for it is a basic property of the arithmetic mean that:The sum of the deviations of the values from the mean is zero.
The question arises:

How will we measure the dispersion that is actually present in our data-set?

Our problem might at first sight seem irresolvable, for by this criterion it appears that no series has any dispersion. Yet we know that this is absolutely incorrect, and we must think of some other way of handling this situation. Surely, we might look at the numerical difference between the mean and the daily fatality figures without considering whether these are positive or negative.
Let us denote these absolute differences by 'modulus of d' or 'mod d'.
This is evident from the third column of the table below:

X	$X - \overline{X} = d$	| d |
4	**0**	**0**
6	**2**	**2**
2	**–2**	**2**
0	**–4**	**4**
3	**–1**	**1**
5	**1**	**1**
8	**4**	**4**
	Total	**14**

By ignoring the sign of the deviations we have achieved a non-zero sum in our second column. Averaging these absolute differences, we obtain a measure of dispersion known as the mean deviation.
In other words, the mean deviation is given by the formula:
MEAN DEVIATION:

$$M.D. = \frac{\sum | d_i |}{n}$$

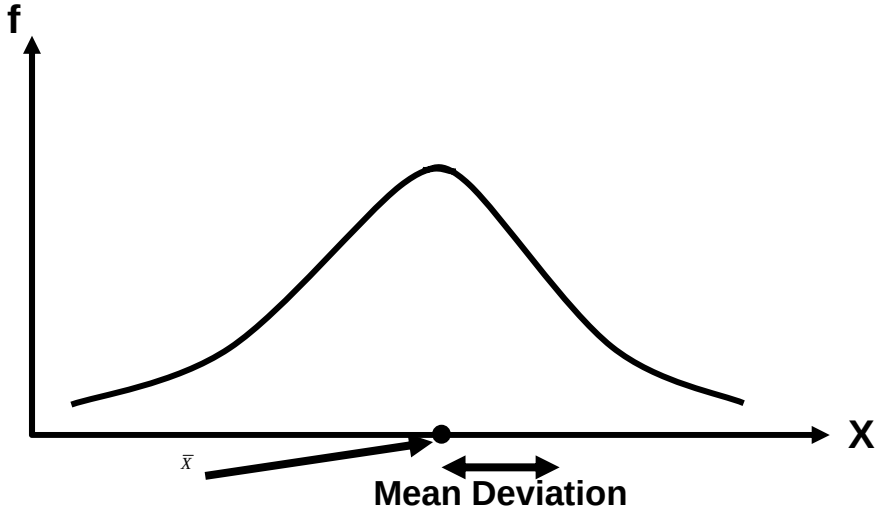
As we are averaging the absolute deviations of the observations from their mean, therefore the complete name of this measure is mean absolute deviation --- but generally we just say "mean deviation". Applying this formula in our example, we find that:
The mean deviation of the number of fatalities is

$$M.D. = \frac{14}{7} = 2.$$

The formula that we have just considered is valid in the case of raw data. In case of grouped data i.e. a frequency distribution, the formula becomes
MEAN DEVIATION FOR GROUPED DATA:

$$M.D. = \frac{\sum f_i \left| x_i - \bar{x} \right|}{n} = \frac{\sum f_i | d_i |}{n}$$

As far as the graphical representation of the mean deviation is concerned, it can be depicted by a horizontal line segment drawn below the X-axis on the graph of the frequency distribution, as shown below:



The approach which we have adopted in the concept of the mean deviation is both quick and simple. But the problem is that we introduce a kind of artificiality in its calculation by ignoring the algebraic signs of the deviations.
In problems involving descriptions and comparisons alone, the mean deviation can validly be applied; but because the negative signs have been discarded, further theoretical development or application of the concept is impossible.

Mean deviation is an absolute measure of dispersion. Its relative measure, known as the co-efficient of mean deviation, is obtained by dividing the mean deviation by the average used in the calculation of deviations i.e. the arithmetic mean. Thus

Co-efficient of M.D:

Sometimes, the mean deviation is computed by averaging the absolute deviations of the data-values from the median i.e.

$$= \frac{M.D.}{Mean}$$

$$\text{Mean deviation} = \frac{\sum |x - \tilde{x}|}{n}$$

And when will we have a situation when we will be using the median instead of the mean?As discussed earlier, the median will be more appropriate than the mean in those cases where our data-set contains a few very high or very low values.In such a situation, the coefficient of mean deviation is given by:

Co-efficient of M.D:

$$= \frac{\text{M.D.}}{\text{Median}}$$

Let us now consider the ***standard deviation*** --- that statistic which is the most important and the most widely used measure of dispersion.

The point that made earlier that from the mathematical point of view, it is not very preferable to take the absolute values of the deviations, *This problem is overcome by computing the standard deviation.*

In order to compute the standard deviation, rather than taking the absolute values of the deviations, we square the deviations.

Averaging these squared deviations, we obtain a statistic that is known as the variance.

Variance

$$= \frac{\sum (x - \bar{x})^2}{n}$$

Let us compute this quantity for the data of the above example.

Our X-values were:

X
4
6
2
0
3
5
8

Taking the deviations of the X-values from their mean, and then squaring these deviations, we obtain:

X	$(x - \bar{x})$	$(x - \bar{x})^2$
4	0	0
6	+ 2	4
2	– 2	4
0	– 4	16
3	– 1	1
5	+ 1	1
8	+ 4	16
		42

Obviously, both (– 2)2 and (2)2 equal 4, both (– 4)2 and (4)2 equal 16, and both (– 1)2 and (1)2 = 1.

Hence $\sum(x - \bar{x})2 = 42$ is now positive, and this positive value has been achieved without 'bending' the rules of mathematics. Averaging these squared deviations, the variance is given by:

Variance:

$$= \frac{\sum(x-\bar{x})^2}{n}$$

$$= \frac{42}{7} = 6$$

The variance is frequently employed in statistical work, but it should be noted that the figure achieved is in 'squared' units of measurement.

In the example that we have just considered, the variance has come out to be "6 squared fatalities", which does not seem to make much sense!

In order to obtain an answer which is in the original unit of measurement, we take the positive square root of the variance. The result is known as the standard deviation.

STANDARD DEVIATION:

$$S = \sqrt{\frac{\sum(x-\bar{x})^2}{n}}$$

Hence, in this example, our standard deviation has come out to be 2.45 fatalities.

In computing the standard deviation (or variance) it can be tedious to first ascertain the arithmetic mean of a series, then subtract it from each value of the variable in the series, and finally to square each deviation and then sum.

It is very much more straight-forward to use the short cut formula given below:

SHORT CUT FORMULA FOR THE STANDARD DEVIATION:

$$S = \sqrt{\left\{ \frac{\sum x^2}{n} - \left(\frac{\sum x}{n} \right)^2 \right\}}$$

In order to apply the short cut formula, we require only the aggregate of the series ($\sum x$) and the aggregate of the squares of the individual values in the series ($\sum x2$).
In other words, only two columns of figures are called for. The number of individual calculations is also considerably reduced, as seen below:

	X	X^2
	4	16
	6	36
	2	4
	0	0
	3	9
	5	25
	8	64
Total	**28**	**154**

Therefore

$$S = \sqrt{\left\{ \frac{154}{7} - \left(\frac{28}{7} \right)^2 \right\}} = \sqrt{(22 - 16)}$$

$$= \sqrt{6} = 2.45 \text{ fatalities}$$

The formulae that we have just discussed are valid in case of raw data. In case of grouped data i.e. a frequency distribution, each squared deviation round the mean must be multiplied by the appropriate frequency figure i.e.

STANDARD DEVIATION IN CASE OF GROUPED DATA:

$$S = \sqrt{\frac{\sum f(x - \bar{x})^2}{n}}$$

And the short cut formula in case of a frequency distribution is:
SHORT CUT FORMULA OF THE STANDARD DEVIATION IN CASE OF GROUPED DATA:

$$S = \sqrt{\left\{ \frac{\sum fx^2}{n} - \left(\frac{\sum fx}{n} \right)^2 \right\}}$$

Which is again preferred from the computational standpoint?

For example, the standard deviation life of a batch of electric light bulbs would be calculated as follows:

EXAMPLE:

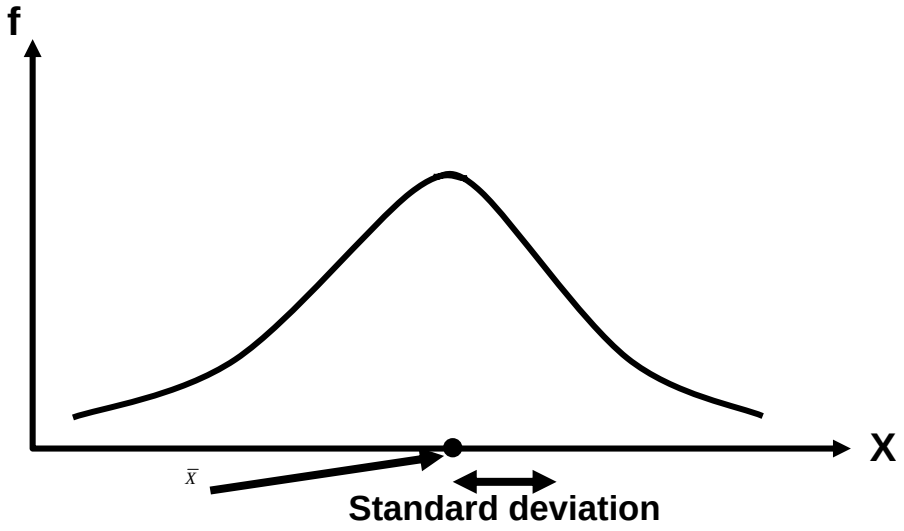
Life (in Hundreds of Hours)	No. of Bulbs f	Mid-point x	fx	fx^2
0 – 5	4	2.5	10.0	25.0
5 – 10	9	7.5	67.5	506.25
10 – 20	38	15.0	570.0	8550.0
20 – 40	33	30.0	990.0	29700.0
40 and over	16	50.0	800.0	40000.0
	100		2437.5	78781.25

Therefore,
standard deviation:

$$S = \sqrt{\left\{\frac{78781.25}{100} - \left(\frac{2437.5}{100}\right)^2\right\}}$$

=13.9hundredhours
= 1390 hours

As far as the graphical representation of the standard deviation is concerned, a horizontal line segment is drawn below the X-axis on the graph of the frequency distribution --- just as in the case of the mean deviation.



The standard deviation is an absolute measure of dispersion. Its relative measure called coefficient of standard deviation is defined as:

Coefficient of S.D:

$$= \frac{\text{Standard Deviation}}{\text{Mean}}$$

And, multiplying this quantity by 100, we obtain a very important and well-known measure called the coefficient of variation.

Coefficient of Variation:

$$\text{C.V.} = \frac{S}{\bar{X}} \times 100$$

As mentioned earlier, the standard deviation is expressed in absolute terms and is given in the same unit of measurement as the variable itself.

There are occasions, however, when this absolute measure of dispersion is inadequate and a relative form becomes preferable.

For example, if a comparison between the variability of distributions with different variables is required, or when we need to compare the dispersion of distributions with the same variable but with very different arithmetic means.

To illustrate the usefulness of the coefficient of variation, let us consider the following two examples:

EXAMPLE-1

Suppose that, in a particular year, the mean weekly earnings of skilled factory workers in one particular country were $ 19.50 with a standard deviation of $ 4, while for its neighboring country the figures were Rs. 75 and Rs. 28 respectively.

From these figures, it is not immediately apparent which country has the GREATER VARIABILITY in earnings.

The coefficient of variation quickly provides the answer:

COEFFICIENT OF VARIATION

For country No. 1:

$$\frac{4}{19.5} \times 100 = 20.5 \text{ per cent,}$$

And for country No. 2:

$$\frac{28}{75} \times 100 = 37.3 \text{ per cent.}$$

From these calculations, it is immediately obvious that the spread of earnings in country No. 2 is greater than that in country No. 1, and the reasons for this could then be sought.

EXAMPLE-2:

The crop yield from 20 acre plots of wheat-land cultivated by ordinary methods averages 35 bushels with a standard deviation of 10 bushels. The yield from similar land treated with a new fertilizer averages 58 bushels, also with a standard deviation of 10 bushels. At first glance, the yield variability may seem to be the same, but in fact it has improved (i.e. decreased) in view of the higher average to which it relates.

Again, the coefficient of variation shows this very clearly:

Coefficient of Variation:

Untreated land:

$$\frac{10}{35} \times 100 = 28.57 \text{ per cent}$$

Treated land:

$$\frac{10}{58} \times 100 = 17.24 \text{ per cent}$$

The coefficient of variation for the untreated land has come out to be 28.57 percent, whereas the coefficient of variation for the treated land is only 17.24 percent.

LECTURE # 29

Permutations and Combinations

- Permutations
- Combinations
- Random Experiment
- Sample Space
- Events
    - Mutually Exclusive Events
    - Exhaustive Events
    - Equally Likely Events

COUNTING RULES:

As discussed in the last lecture, there are certain rules that facilitate the calculations of probabilities in certain situations. They are known as counting rules and include concepts of:

1) **Multiple Choice**
2) **Permutations**
3) **Combinations**

We have already discussed the rule of multiplication in the last lecture.
Let us now consider the rule of permutations.

RULE OF PERMUTATION:

A permutation is any ordered subset from a set of n distinct objects.
For example, if we have the set {a, b}, then one permutation is ab, and the other permutation is ba.
The number of permutations of r objects, selected in a definite order from n distinct objects is denoted by the symbol nPr, and is given by
nPr = n (n – 1) (n – 2) …(n – r + 1)

$$= \frac{n!}{(n-r)!}.$$

FACTORIALS

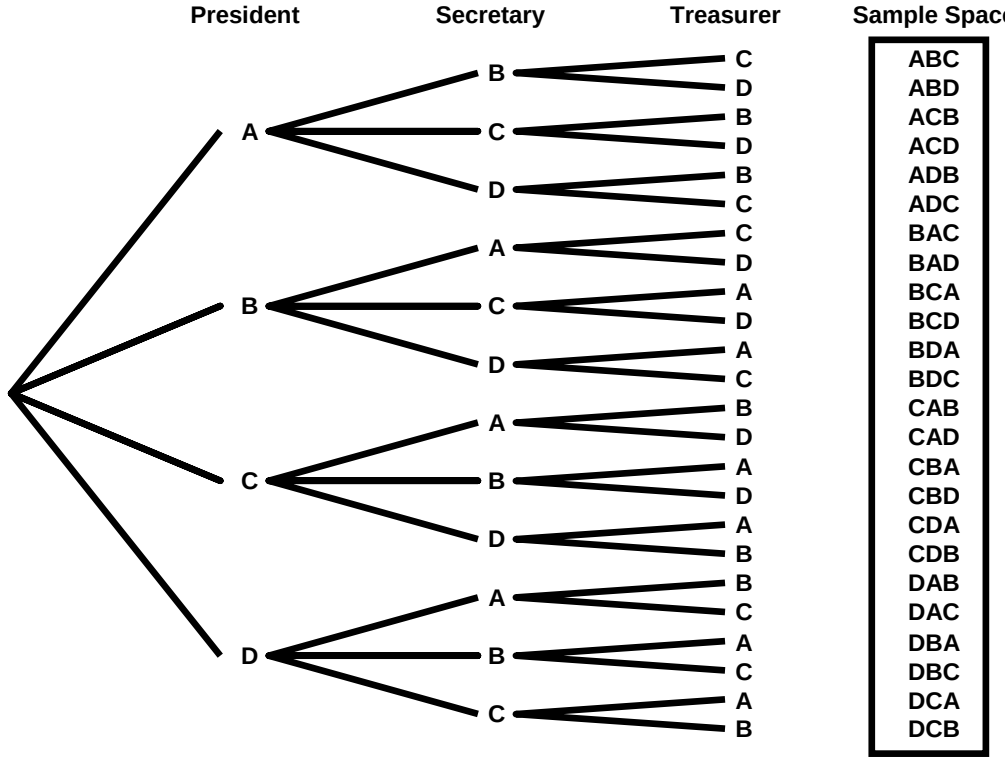
7! = 7 × 6 × 5 × 4 × 3 × 2 × 1
6! = 6 × 5 × 4 × 3 × 2 × 1
.
.
.
1! = 1

Also, we define 0! = 1.

Example:
A club consists of four members. How many ways are there of selecting three officers: president, secretary and treasurer?
It is evident that the order in which 3 officers are to be chosen, is of significance.
Thus there are 4 choices for the first office, 3 choices for the second office, and 2 choices for the third office. Hence the total number of ways in which the three offices can be filled is 4 × 3 × 2 = 24.
The same result is obtained by applying the rule of permutations:

$$\begin{aligned} {}^4P_3 &= \frac{4!}{(4-3)!} \\ &= 4 \times 3 \times 2 \\ &= 24 \end{aligned}$$

Let the four members be, A, B, C and D. Then a tree diagram which provides an organized way of listing the possible arrangements, for this example, is given below:



PERMUTATIONS:

In the formula of nPr, if we put r = n, we obtain:

$nPn = n(n-1)(n-2) \ldots 3 \times 2 \times 1$

$= n!$

I.e. the total number of permutations of n distinct objects, taking all n at a time, is equal to n!

EXAMPLE:

Suppose that there are three persons A, B & D, and that they wish to have a photograph taken. The total number of ways in which they can be seated on three chairs (placed side by side) is

$3P3 = 3! = 6$

These are:

ABD,
ADB,
BAD,
BDA,
DAB,
& DBA.

The above discussion pertained to the case when all the objects under consideration are distinct objects. If some of the objects are not distinct, the formula of permutations modifies as given below:

The number of permutations of n objects, selected all at a time, when n objects consist of n1 of one kind, n2 of a second kind, …, nk of a kth kind,

$$\text{is } P = \frac{n!}{n_1!\, n_2! \ldots .. n_k!}.$$

$$\left(\text{where } \sum n_i = n\right)$$

EXAMPLE:
How many different (meaningless) words can be formed from the word 'committee'?
In this example:
n = 9 (because the total number of letters in this word is 9)
n1 = 1 (because there is one c)
n2 = 1 (because there is one o)
n3 = 2 (because there are two m's)
n4 = 1 (because there is one i)
n5 = 2 (because there are two t's)
and
n6 = 2 (because there are two e's)
Hence, the total number of (meaningless) words
(permutations) is:

$$P = \frac{n!}{n_1!\ n_2!\ .....\ n_k!}.$$
$$= \frac{9!}{1!\ \ 1!\ \ 2!\ \ 1!\ \ 2!\ \ 2!}$$
$$= \frac{9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{1\times 1\times 2\times 1\times 1\times 2\times 1\times 2\times 1}$$
$$= 45360$$

Next, let us consider the rule of combinations.
RULE OF COMBINATION:
A combination is any subset of r objects, selected without regard to their order, from a set of n distinct objects. The total number of such combinations is denoted by the symbol

$$^{n}C_{r} \ or \ \binom{n}{r},$$

and is given by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

where r < n.
It should be noted that

$$^{n}P_{r} = r!\binom{n}{r}$$

In other words, every combination of r objects (out of n objects) generates r! Permutations.
EXAMPLE:
Suppose we have a group of three persons, A, B, & C. If we wish to select a group of two persons out of these three, the three possible groups are {A, B}, {A, C} and {B, C}.In other words, the total number of combinations of size two out of this set of size three is 3.

Now, suppose that our interest lies in forming a committee of two persons, one of whom is to be the president and the other the secretary of a club.
The six possible committees are:

(A, B) , (B, A),
(A, C) , (C, A),
(B, C) & (C, B).

In other words, the total number of permutations of two persons out of three is 6.
And the point to note is that each of three combinations mentioned earlier generates 2 = 2! permutations.

i.e. the combination {A, B} generates the permutations
(A, B) and (B, A);
the combination {A, C} generates the permutations
(A, C) and (C, A); and
the combination {B, C} generates the permutations
(B, C) and (C, B).

The quantity

$$\binom{n}{r}$$

or nCr is also called a binomial co-efficient because of its appearance in the binomial expansion of

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}\ b^r .$$

The binomial co-efficient has two important properties.

i) $$\binom{n}{r} = \binom{n}{n-r}, \text{ and}$$

ii) $$\binom{n}{n-r} + \binom{n}{r} = \binom{n+1}{r}$$

Also, it should be noted that

$$\binom{n}{0} = 1 = \binom{n}{n}$$

and

$$\binom{n}{1} = n = \binom{n}{n-1}$$

EXAMPLE:
A three-person committee is to be formed out of a group of ten persons. In how many ways can this be done?
Since the order in which the three persons of the committee are chosen, is unimportant, it is therefore an example of a problem involving combinations. Thus the desired number of combinations is

$$\binom{n}{r} = \binom{10}{3} = \frac{10!}{3!\,(10-3)!} = \frac{10!}{3!\,7!}$$
$$= \frac{10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{3\times 2\times 1\times 7\times 6\times 5\times 4\times 3\times 2\times 1}$$
$$= 120$$

In other words, there are one hundred and twenty different ways of forming a three-person committee out of a group of only ten persons!

EXAMPLE:

In how many ways can a person draw a hand of 5 cards from a well-shuffled ordinary deck of 52 cards?

The total number of ways of doing so is given by

$$\binom{n}{r} = \binom{52}{5} = \frac{52\times 51\times 50\times 49\times 48}{5\times 4\times 3\times 2\times 1} = 2{,}598{,}960$$

Having reviewed the counting rules that facilitate calculations of probabilities in a number of problems, let us now begin the discussion of concepts that lead to the formal definitions of probability.

The first concept in this regard is the concept of Random Experiment.

The term experiment means a planned activity or process whose results yield a set of data. A single performance of an experiment is called a trial. The result obtained from an experiment or a trial is called an outcome.

RANDOM EXPERIMENT:

An experiment which produces different results even though it is repeated a large number of times under essentially similar conditions is called a Random Experiment.

The tossing of a fair coin, the throwing of a balanced die, drawing of a card from a well-shuffled deck of 52 playing cards, selecting a sample, etc. are examples of random experiments.

A random experiment has three properties:

i) The experiment can be repeated, practically or theoretically, any number of times.

ii) The experiment always has two or more possible outcomes. An experiment that has only one possible outcome, is not a random experiment.

iii) The outcome of each repetition is unpredictable, i.e. it has some degree of uncertainty.

Considering a more realistic example, interviewing a person to find out whether or not he or she is a smoker is an example of a random experiment. This is so because this example fulfils all the three properties that have just been discussed:

1.This process of interviewing can be repeated a large number of times.

2.To each interview, there are at least two possible replies: 'I am a smoker' and 'I am not a smoker'.

3.For any interview, the answer is not known in advance i.e. there is an element of uncertainty regarding the person's reply.

A concept that is closely related with the concept of a random experiment is the concept of the Sample Space.

SAMPLE SPACE:

A set consisting of all possible outcomes that can result from a random experiment (real or conceptual), can be defined as the sample space for the experiment and is denoted by the letter S.

Each possible outcome is a member of the sample space, and is called a sample point in that space.

Let us consider a few examples:

EXAMPLE-1

The experiment of tossing a coin results in either of the two possible outcomes: a head (H) or a tail (T).
(We assume that it is not possible for the coin to land on its edge or to roll away.)
The sample space for this experiment may be expressed in set notation as S = {H, T}.
'H' and 'T' are the two sample points.

EXAMPLE-2

The sample space for tossing two coins once (or tossing a coin twice) will contain four possible outcomes denoted by

S = {HH, HT, TH, TT}.

In this example, clearly, S is the Cartesian product A × A, where A = {H, T}.

EXAMPLE-3

The sample space S for the random experiment of throwing two six-sided dice can be described by the Cartesian product A × A, where
A = {1, 2, 3, 4, 5,6}.
In other words,
$S = A \times A$
$= \{(x, y) \mid x \in A \text{ and } y \in A\}$
where x denotes the number of dots on the upper face of the first die, and y denotes the number of dots on the upper face of the second die.
Hence, S contains 36 outcomes or sample points, as shown below:

S = { (1, 1), (1, 2), (1, 3), (1, 5), (1, 6),
(2, 1), (2, 2), (2, 3), (2, 5), (2, 6),
(3, 1), (3, 2), (3, 3), (3, 5), (3, 6),
(4, 1), (4, 2), (4, 3), (4, 5), (4, 6),
(5, 1), (5, 2), (5, 3), (5, 5), (5, 6),
(6, 1), (6, 2), (6, 3), (6, 5), (6, 6) }

The next concept is that of events:

EVENTS:

Any subset of a sample space S of a random experiment, is called an event.
In other words, an event is an individual outcome or any number of outcomes (sample points) of a random experiment.

SIMPLE & COMPOUND EVENTS:

An event that contains exactly one sample point, is defined as a simple event. A compound event contains more than one sample point, and is produced by the union of simple events.

EXAMPLE

The occurrence of a 6 when a die is thrown, is a simple event, while the occurrence of a sum of 10 with a pair of dice, is a compound event, as it can be decomposed into three simple events (4, 6), (5, 5) and (6, 4).

OCCURRENCE OF AN EVENT:

An event A is said to occur if and only if the outcome of the experiment corresponds to some element of A.

EXAMPLE:

Suppose we toss a die, and we are interested in the occurrence of an even number.
If ANY of the three numbers '2', '4' or '6' occurs, we say that the event of our interest has occurred.
In this example, the event A is represented by the set {2, 4, 6}, and if the outcome '2' occurs, then, since this outcome is corresponding to the first element of the set A, therefore, we say that A has occurred.

COMPLEMENTARY EVENT:

The event "not-A" is denoted by $\bar{A}$ or Ac and called the negation (or complementary event) of A.

EXAMPLE:

If we toss a coin once, then the complement of "heads" is "tails".

If we toss a coin four times, then the complement of "at least one head" is "no heads".

A sample space consisting of n sample points can produce 2n different subsets (or simple and compound events).

EXAMPLE:

Consider a sample space S containing 3 sample points, i.e. S = {a, b, c}.

Then the 23 = 8 possible subsets are

ϕ, {a}, {b}, {c}, {a, b},
{a, c}, {b, c}, {a, b, c}

Each of these subsets is an event.

The subset {a, b, c} is the sample space itself and is also an event. It always occurs and is known as the certain or sure event.

The empty set ϕ is also an event, sometimes known as impossible event, because it can never occur.

MUTUALLY EXCLUSIVE EVENTS:

Two events A and B of a single experiment are said to be mutually exclusive or disjoint if and only if they cannot both occur at the same time i.e. they have no points in common.

EXAMPLE-1:

When we toss a coin, we get *either* a head *or* a tail, but *not* both at the same time.

The two events head and tail are therefore mutually exclusive.

EXAMPLE-2:

When a die is rolled, the events 'even number' and 'odd number' are mutually exclusive as we can get either an even number or an odd number in one throw, not both at the same time. Similarly, a student *either* qualifies or fails, a single birth must be *either* a boy or a girl, it cannot be both, etc., etc. Three or more events originating from the same experiment are mutually exclusive if pair wise they are mutually exclusive. If the two events *can* occur at the same time, they are not mutually exclusive, e.g., if we draw a card from an ordinary deck of 52 playing cars, it *can* be both a king and a diamond.

Therefore, kings and diamonds are not mutually exclusive.

Similarly, inflation and recession are not mutually exclusive events.

Speaking of playing cards, it is to be remembered that an ordinary deck of playing cards contains 52 cards arranged in 4 suits of 13 each. The four suits are called diamonds, hearts, clubs and spades; the first two are red and the last two are black.

The face values called denominations, of the 13 cards in each suit are ace, 2, 3, …, 10, jack, queen and king.

The *face cards* are king, queen and jack.

These cards are used for various games such as whist, bridge, poker, etc.

We have discussed the concepts of mutually exclusive events.

Another important concept is that of exhaustive events.

EXHAUSTIVE EVENTS:

Events are said to be collectively exhaustive, when the union of mutually exclusive events is equal to the entire sample space S.

EXAMPLES:

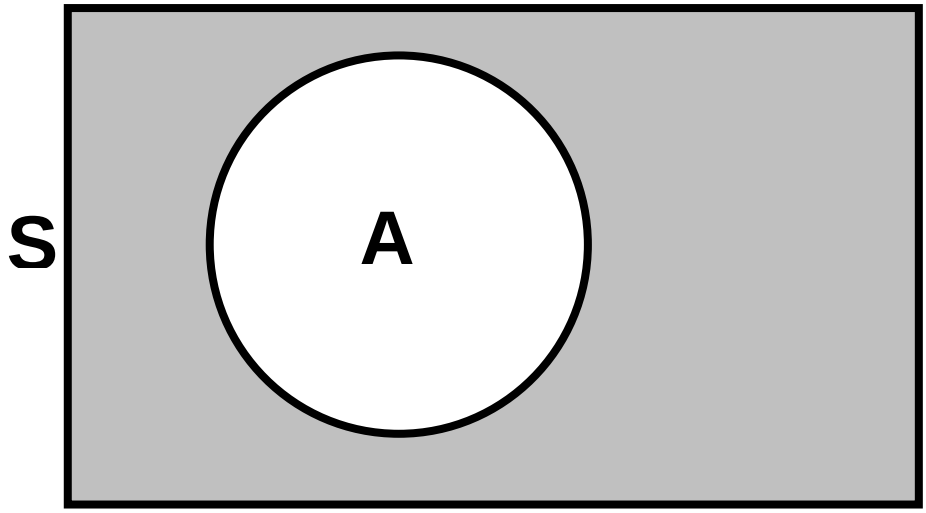
1. In the coin-tossing experiment, 'head' and 'tail' are collectively exhaustive events.

2. In the die-tossing experiment, 'even number' and 'odd number' are collectively exhaustive events.

In conformity with what was discussed in the last lecture:

PARTITION OF THE SAMPLE SPACE:

A group of mutually exclusive and exhaustive events belonging to a sample space is called a partition of the sample space. With reference to any sample space S, events A and $\bar{A}$ form a partition as they are mutually exclusive and their union is the entire sample space. The Venn Diagram below clearly indicates this point.



$\bar{A}$ is shaded

Next, we consider the concept of equally likely events:

EQUALLY LIKELY EVENTS:

Two events A and B are said to be equally likely, when one event is as likely to occur as the other. In other words, each event should occur in equal number in repeated trials.

EXAMPLE:

When a fair coin is tossed, the head is as likely to appear as the tail, and the proportion of times each side is expected to appear is 1/2.

EXAMPLE:

If a card is drawn out of a deck of well-shuffled cards, each card is equally likely to be drawn, and the probability that any card will be drawn is 1/52.

LECTURE # 30

Definitions of Probability

Definitions of Probability:

- Subjective Approach to Probability
- Objective Approach:
- Classical Definition of Probability

Relative Frequency Definition of Probability

Before we begin the various definitions of probability, let us revise the concepts of:

- Mutually Exclusive Events
- Exhaustive Events
- Equally Likely Events

MUTUALLY EXCLUSIVE EVENTS:

Two events A and B of a single experiment are said to be mutually exclusive or disjoint if and only if they cannot both occur at the same time i.e. they have no points in common.

EXAMPLE-1:

When we toss a coin, we get *either* a head *or* a tail, but *not* both at the same time.

The two events head and tail are therefore mutually exclusive.

EXAMPLE-2:

When a die is rolled, the events 'even number' and 'odd number' are mutually exclusive as we can get either an even number or an odd number in one throw, not both at the same time. Similarly, a student *either* qualifies *or* fails, a person is either a teenager or not a teenager, etc., etc.

Three or more events originating from the same experiment are mutually exclusive if pair wise they are mutually exclusive.

If the two events *can* occur at the same time, they are not mutually exclusive, e.g., if we draw a card from an ordinary deck of 52 playing cars, it *can* be both a king and a diamond.

Therefore, kings and diamonds are not mutually exclusive. Speaking of playing cards, it is to be remembered that an ordinary deck of playing cards contains 52 cards arranged in 4 suits of 13 each. The four suits are called diamonds, hearts, clubs and spades; the first two are red and the last two are black. The face values called denominations, of the 13 cards in each suit are ace, 2, 3, …, 10, jack, queen and king. The face values called denominations, of the 13 cards in each suit are ace, 2, 3, …, 10, jack, queen and king.

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The Venn D

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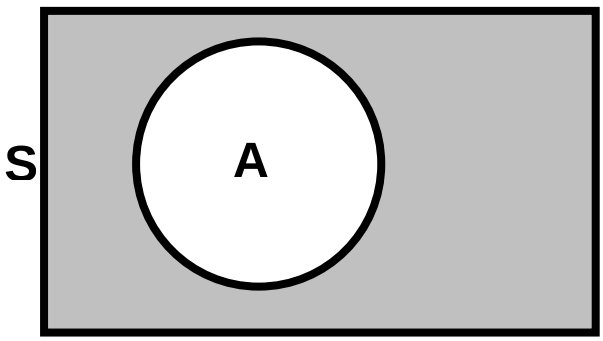
Venn Diagram



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Next, we consider the concept of equally likely events:

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Two events A and B are said to be equally likely, when one event is as likely to occur as the other.

In other words, each event should occur in equal number in repeated trials.

EXAMPLE:

When a fair coin is tossed, the head is as likely to appear as the tail, and the proportion of times each side is expected to appear is 1/2.

EXAMPLE:

If a card is drawn out of a deck of well-shuffled cards, each card is equally likely to be drawn, and the proportion of times each card can be expected to be drawn in a *very* large number of draws is 1/52.Having discussed basic concepts related to probability theory, we now begin the discussion of THE CONCEPT AND DEFINITIONS OF PROBABILITY.

Probability can be discussed from two points of view: the subjective approach, and the objective approach.

SUBJECTIVE OR PERSONALISTIC PROBABILITY:

As its name suggests, the subjective or personalistic probability is a measure of the strength of a person's belief regarding the occurrence of an event A. Probability in this sense is purely subjective, and is based on whatever evidence is available to the individual. It has a disadvantage that two or more persons faced with the same evidence may arrive at different probabilities.

For example, suppose that a panel of three judges is hearing a trial. It is possible that, based on the evidence that is presented, two of them arrive at the conclusion that the accused is guilty while one of them decides that the evidence is NOT strong enough to draw this conclusion.

On the other hand, **objective probability** relates to those situations where everyone will arrive at the same conclusion.

It can be classified into two broad categories, each of which is briefly described as follows:

1. The Classical or 'A Priori' Definition of Probability

If a random experiment can produce n mutually exclusive and equally likely outcomes, and if m out to these outcomes are considered favorable to the occurrence of a certain event A, then the probability of the event A, denoted by P(A), is defined as the ratio m/n.

Symbolically, we write

$$P(A) = \frac{m}{n}$$

$$= \frac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}}$$

This definition was formulated by the French mathematician P.S. Laplace (1949-1827) and can be very conveniently used in experiments where the total number of possible outcomes and the number of outcomes favourable to an event can be DETERMINED.

Let us now consider a few examples to illustrate the classical definition of probability:

EXAMPLE-1:

If a card is drawn from an ordinary deck of 52 playing cards, find the probability that
i) the card is a red card, ii) the card is a 10.

SOLUTION :

The total number of possible outcomes is 13+13+13+13 = 52, and we assume that all possible outcomes are equally likely.(It is well-known that an ordinary deck of cards contains 13 cards of diamonds, 13 cards of hearts, 13 cards of clubs, and 13 cards of spades.)

(i) Let A represent the event that the card drawn is a red card.

Then the number of outcomes favourable to the event A is 26 (since the 13 cards of diamonds and the 13 cards of hearts are *red*).

Hence

$$P(A) = \frac{m}{n}$$

$$= \frac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}}$$

$$= \frac{26}{52} = \frac{1}{2}$$

EXAMPLE-2:

Thus $P(B) = \frac{4}{52} = \frac{1}{13}$.

A fair coin is tossed three times. What is the probability that at least one head appears?

SOLUTION:

The sample space for this experiment is

S = {HHH, HHT, HTH, THH,
HTT, THT, TTH, TTT}

and thus the total number of sample points is 8

i.e. n(S) = 8.Let A denote the event that at least one head appears. Then

A = {HHH, HHT, HTH,
THH, HTT, THT, TTH}

and therefore n(A) = 7.

Hence

$$P(A) = \frac{n(A)}{n(S)} = \frac{7}{8}.$$

EXAMPLE-3:

Four items are taken at random from a box of 12 items and inspected. The box is *rejected* if more than 1 item is found to be faulty. If there are 3 faulty items in the box, find the probability that the box is *accepted*.

SOLUTION:

The sample space S contains $\binom{12}{4} = 495$

sample points

(because there are $\binom{12}{4}$

ways of selecting four items out of twelve).

The box contains 3 faulty and 9 good items. The box is accepted if there is (i) no faulty items, or (ii) one faulty item in the sample of 4 items *selected*.
Let A denote the event the number of faulty items chosen is 0 or 1.
Then

$$n(A) = \binom{3}{0}\binom{9}{4} + \binom{3}{1}\binom{9}{3}$$

$$= 126 + 252 = 378 \ \ sample \ \ points.$$

$$\therefore \quad P(A) = \frac{m}{n} = \frac{378}{495} = 0.76$$

Hence the probability that the box is accepted is 76% (in spite of the fact that the box *contains* 3 faulty items).

The classical definition has the following shortcomings:

i) This definition is said to involve circular reasoning as the term equally likely really means equally probable.
Thus probability is defined by introducing concepts that presume a *prior* knowledge of the *meaning* of probability.
ii) This definition becomes vague when the possible outcomes are INFINITE in number, or uncountable.
iii) This definition is NOT applicable when the assumption of equally likely does *not* hold. And the fact of the matter is that there are NUMEROUS situations where the assumption of equally likely cannot hold.
And these are the situations where we have to look for another definition of probability!

The *other* definition of probability under the objective approach is the **relative frequency definition of probability.**

The essence of this definition is that if an experiment is repeated a large number of times under (more or less) identical conditions, and if the event of our interest occurs a certain number of times, then the *proportion* in which this event occurs is regarded as the probability of that event.

For example, we know that a large number of students sit for the matric examination every year. Also, we know that a certain proportion of these students will obtain the first division, a certain proportion will obtain the second division, --- and a certain proportion of the students will fail.
Since the total number of students appearing for the matric exam is very large, hence:

- The proportion of students who obtain the first division --- this proportion can be regarded as the *probability* of obtaining the first division,
- The proportion of students who obtain the second division --- this proportion can be regarded as the *probability* of obtaining the second division, and so on.

LECTURE # 31

Relative Frequency, Axiomatic Definition & Laws of Probability

- Relative Frequency Definition of Probability
- Axiomatic Definition of Probability
- Laws of Probability
- Rule of Complementation
- Addition Theorem

THE RELATIVE FREQUENCY DEFINITION OF PROBABILITY
('A POSTERIORI' DEFINITION OF PROBABILITY):

If a random experiment is repeated a large number of times, say n times, under identical conditions and if an event A is observed to occur m times, then the probability of the event A is defined as the LIMIT of the relative frequency m/n as n tends to infinitely.
Symbolically, we write

$$P(A) = \lim_{n\to\infty} \frac{m}{n}$$

The definition assumes that as n increases indefinitely, the ratio m/n tends to become stable at the numerical value P(A). The relationship between relative frequency and probability can also be represented as follows:

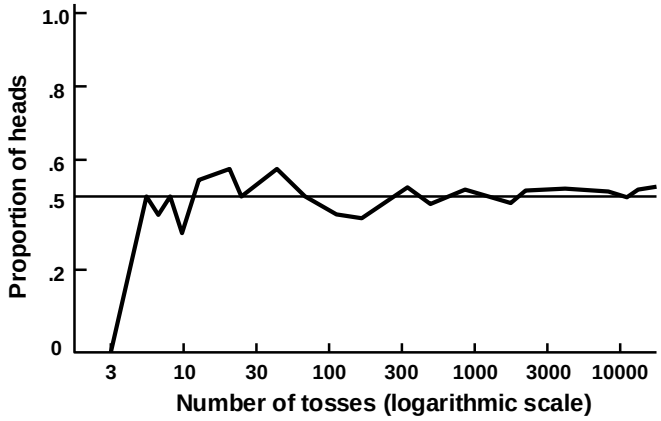
Relative Frequency $\rightarrow$ Probability
as n $\rightarrow \infty$

As its name suggests, the relative frequency definition relates to the relative frequency with which are event occurs in the *long run*. In situations where we can say that an experiment has been repeated a very large number of times, the *relative frequency definition* can be applied.

As such, this definition is very useful in those practical situations where we are interested in computing a probability in numerical form but where the classical definition cannot be applied.(Numerous real-life situations are such where various possible outcomes of an experiment are NOT equally likely). This type of probability is also called empirical probability as it is based on *EMPIRICAL* evidence i.e. on *OBSERVATIONAL* data.
It can also be called ***STATISTICAL PROBABILITY*** for it is this very probability that forms the basis of mathematical statistics.
Let us try to understand this concept by means of two examples:
1) from a coin-tossing experiment and
2) from data on the numbers of boys and girls born.

EXAMPLE-1:
Coin-Tossing:
No one can tell which way a coin will fall but we expect the proportion of leads and tails after a large no. of tosses to be nearly equal. An experiment to demonstrate this point was performed by Kerrich in Denmark in 1946. He tossed a coin 10,000 times, and obtained altogether 5067 heads and 4933 tails.
The behavior of the proportion of heads throughout the experiment is shown as in the following figure:

The proportion; of heads in a sequence of tosses of a coin (Kerrich, 1946):



As you can see, the curve fluctuates *widely* at first, but begins to settle down to a more or less *stable* value as the number of spins increases. It seems reasonable to suppose that the fluctuations would continue to diminish if the experiment were continued *indefinitely*, and the proportion of heads would cluster more and more *closely* about a *limiting* value which would be *very near*, if not exactly, one-half.

This hypothetical *limiting* value is the (statistical) probability of heads.

Let us now take an example closely related to our *daily* lives --- that relating to the sex ratio:-

In this context, the first point to note is that it has been known since the eighteenth century that in *reliable* birth statistics based on sufficiently *large* numbers (in at least some parts of the world), there is always a slight *excess* of boys,

Laplace records that, among the 215,599 births in thirty districts of France in the years 1800 to 1802, there were 110,312 boys and 105,287 girls.

The proportions of boys and girls were thus 0.512 and 0.488 respectively (indicating a slight *excess* of boys over girls).In a *smaller* number of births one would, however, expect considerable *deviations* from these proportions.

This point can be illustrated with the help of the following example:

EXAMPLE-2:

The following table shows the proportions of male births that have been worked out for the major regions of England as well as the rural districts of Dorset (for the year 1956):

Proportions of Male Births in various Regions and Rural Districts of England in 1956

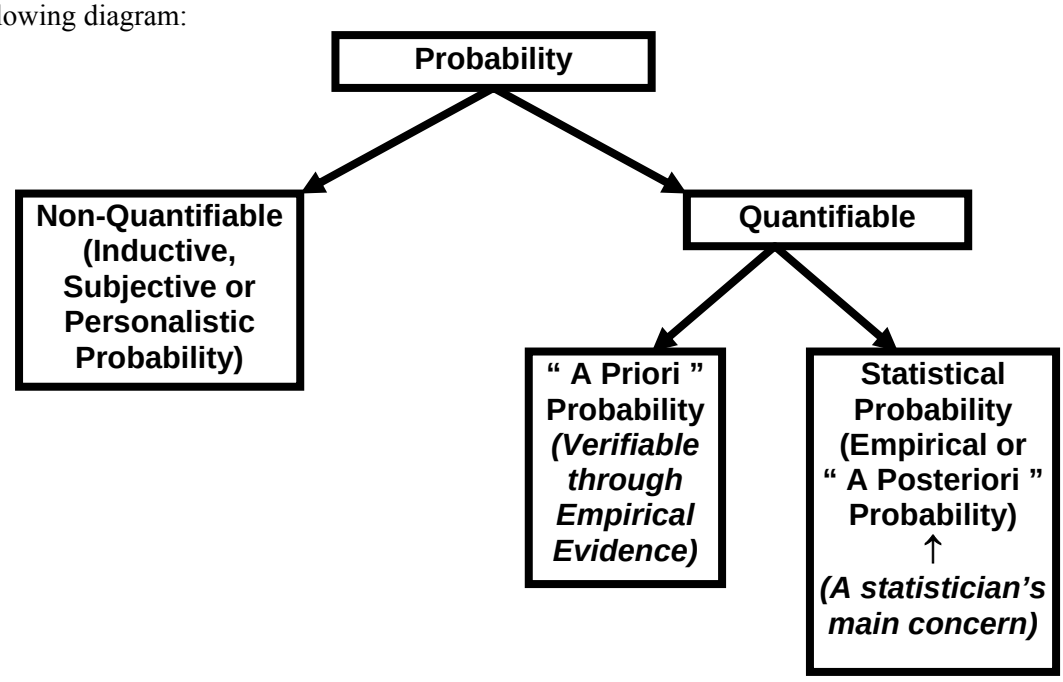
(Source: Annual Statistical Review)

Region of England	Proportion of Male Births	Rural Districts of Dorset	Proportion of Male Births
Northern	.514	Beaminster	.38
E. & W. Riding	.513	Blandford	.47
North Western	.512	Bridport	.53
North Midland	.517	Dorchester	.50
Midland	.514	Shaftesbury	.59
Eastern	.516	Sherborne	.44
London and S. Eastern	.514	Sturminster	.54
Southern	.514	Wareham and Purbeck	.53
South Western	.513	Wimborne & Cranborne	.54
Whole country	.514	All Rural District's of Dorset	.512

As you can see, the figures for the rural districts of Dorset, based on about 200 births each, fluctuate between 0.38 and 0.59. While those for the major regions of England, which are each based on about 100,000 births, do not fluctuate much, rather, they range between 0.512 and 0.517 only. The larger sample size is clearly the reason for the greater constancy of the latter. We can imagine that if the sample were increased indefinitely, the proportion of boys would tend to a *limiting* value which is unlikely to differ much from 0.514, the proportion of male births for the *whole* country.

This hypothetical *limiting* value is the (statistical) *probability* of a male birth.

The overall discussion regarding the various ways in which probability can be defined is presented in the following diagram:



As far as *quantifiable* probability is concerned, in those situations where the various possible outcomes of our experiment are equally likely, we can compute the probability *prior* to actually

conducting the experiment --- otherwise, as is generally the case, we can compute a probability only *after* the experiment has been conducted (and this is why it is also called 'a posteriori' probability).

***Non*-quantifiable probability is the one that is called Inductive Probability.**

It refers to the degree of belief which it is reasonable to place in a proposition on *given* evidence.

An important point to be noted is that it is difficult to express inductive probabilities numerically — to construct a numerical scale of inductive probabilities, with 0 standing for impossibility and for logical certainty. An important point to be noted is that it is difficult to express inductive probabilities numerically — to construct a numerical scale of inductive probabilities, with 0 standing for impossibility and for logical certainty.

Most statisticians have arrived at the conclusion that inductive probability cannot, in general, he measured and, therefore cannot be use in the mathematical theory of statistics.

This conclusion is not, perhaps, very surprising since there seems no reason why rational degree of belief should be measurable any more than, say, degrees of beauty. Some paintings are very beautiful, some are quite beautiful, and some are ugly, but it would be observed to try to construct a numerical scale of beauty, on which Mona Lisa had a beauty value of 0.96.Similarly some propositions are highly probable, some are quite probable and some are improbable, but it does not seem possible to construct a numerical scale of such (inductive) probabilities .Because of the fact that inductive probabilities are not quantifiable and cannot be employed in a mathematical argument, this is the reason why the usual methods of statistical inference such as tests of significance and confidence interval are based entirely on the concept of statistical probability. Although we have discussed three different ways of defining probability, the most formal definition is yet to come.

This is The Axiomatic Definition of Probability.

THE AXIOMATIC DEFINITION OF PROBABILITY:

This definition, introduced in 1933 by the Russian mathematician Andrei N. Kolmogrov, is based on a set of AXIOMS.

Let S be a sample space with the sample points E1, E2, … Ei, …En. To each sample point, we assign a real number, denoted by the symbol P(Ei), and called the probability of Ei, that must satisfy the following basic axioms:

Axiom 1:

For any event Ei,
$0 < P(Ei) < 1$.

Axiom 2:

$P(S) = 1$
for the sure event S.

Axiom 3:

If A and B are mutually exclusive events (subsets of S), then

$P(A \cup B) = P(A) + P(B)$.

It is to be emphasized that According to the axiomatic theory of probability:

SOME probability defined as a non-negative real number is to be ATTACHED to each sample point Ei *such that the sum of all such numbers must equal ONE*.

The ASSIGNMENT of probabilities may be based on past evidence or on some other underlying conditions.

(If this assignment of probabilities is based on past evidence, we are talking about EMPIRICAL probability, and if this assignment is based on underlying conditions that ensure that the various possible outcomes of a random experiment are EQUALLY LIKELY, then we are talking about the CLASSICAL definition of probability.

Let us consider another example:

EXAMPLE :

Table-1

below shows the numbers of births in England and Wales in 1956 classified by (a) sex and (b) whether liveborn or stillborn.

Table-1

Number of births in England and Wales in 1956 by sex and whether live- or still born. (Source Annual Statistical Review)

	Liveborn	Stillborn	Total
Male	359,881 (A)	8,609 (B)	368,490
Female	340,454 (B)	7,796 (D)	348,250
Total	700,335	16,405	716,740

There are four possible events in this double classification:

- Male livebirth (denoted by A),
- Male stillbirth (denoted by B),
- Female livebirth (denoted by C)

and

- Female stillbirth (denoted by D),

The relative frequencies corresponding to the figures of Table-1 are given in Table-2:

Table-2

Proportion of births in England and Wales in 1956 by sex and whether live- or stillborn. (Source *Annual Statistical Review*)

	Liveborn	Stillborn	Total
Male	.5021	.0120	.5141
Female	.4750	.0109	.4859
Total	.9771	.0229	1.0000

The total number of births is large enough for these relative frequencies to be treated for all practical purposes as *PROBABILITIES*.

Let us denote the compound events 'Male birth' and 'Stillbirth' by the letters M and S.Now a male birth occurs whenever either a male livebirth or a male stillbirth occurs, and so the proportion of male birth, regardless of whether they are live-or stillborn, is equal to the sum of the proportions of these two types of birth; that is to say,

$p(M) = p(A \text{ or } B) = p(A) + p(B)$

$= .5021 + .0120 = .5141$

Similarly, a stillbirth occurs whenever either a male stillbirth or a female stillbirth occurs and so the proportion of stillbirths, regardless of sex, is equal to the sum of the proportions of these two events:

$p(S) = p(B \text{ or } D) = p(B) + p(D)$

$= .0120 + .0109 = .0229.$

Let us now consider some basic LAWS of probability.

These laws have important applications in solving probability problems.

LAW OF COMPLEMENTATION:

If $\overline{A}$ is the complement of an event A relative to the sample space S, then

$$P(\overline{A}) = 1 - P(A).$$

Hence the probability of the complement of an event is equal to one minus the probability of the event.

Complementary probabilities are very useful when we are wanting to solve questions of the type 'What is the probability that, in tossing two fair dice, at least one even number will appear?'

EXAMPLE:

A coin is tossed 4 times in succession. What is the probability that at least one head occurs?

(1) The sample space S for this experiment consists of 24 = 16 sample points (as each toss can result in 2 outcomes),and

(2) we assume that each outcome is equally likely.

If we let A represent the event that at least one head occurs, then A will consist of MANY sample points, and the process of computing the probability of this event will become somewhat cumbersome! So, instead of denoting this particular event by A, let us denote its complement i.e. "No head" by A.

Thus the event A consists of the SINGLE sample point {TTTT}.

Therefore P(A) = 1/16.

Hence by the law of complementation, we have

$$P(\overline{A}) = 1 - P(A) = 1 - \frac{1}{16} = \frac{15}{16}.$$

The next law that we will consider is the Addition Law or the General Addition Theorem of Probability:

ADDITION LAW:

If A and B are any two events defined in a sample space S, then

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

In words, this law may be stated as follows:

"If two events A and B are not mutually exclusive, then the probability that at least one of them occurs, is given by the sum of the separate probabilities of events A and B minus the probability of the joint event $A \cap B$.**"**

LECTURE # 32

Independent and Dependent Events, Multiplication Theorem & Marginal Probability

- Independent and Dependent Events
- Multiplication Theorem of Probability for Independent Events
- Marginal Probability

Before we proceed the concept of independent versus dependent events, let us review the Addition and Multiplication Theorems of Probability that were discussed in the last lecture.

To this end, let us consider an interesting example that illustrates the application of both of these theorems in one problem:

EXAMPLE:

A bag contains 10 white and 3 black balls. Another bag contains 3 white and 5 black balls. Two balls are transferred from first bag and placed in the second, and then one ball is taken from the latter.

What is the probability that it is a white ball?

In the beginning of the experiment, we have:

Colour of Ball	No. of Balls in Bag A	No. of Balls in Bag B
White	10	3
Black	3	5
Total	13	8

Let A represent the event that 2 balls are drawn from the first bag and transferred to the second bag. Then A can occur in the following three mutually exclusive ways:

A1 = 2 white balls are transferred to the second bag.

A2 = 1 white ball and 1 black ball are transferred to the second bag.

A3 = 2 black balls are transferred to the second bag.

Then, the total number of ways in which 2 balls can be drawn out of a total of 13 balls is $\binom{13}{2}$.

And, the total number of ways in which 2 white balls can be drawn out of 10 white balls is $\binom{10}{2}$.

Thus, the probability that two white balls are selected from the first bag containing 13 balls (in order to *transfer* to the second bag) is

$$P(A_1) = \binom{10}{2} \div \binom{13}{2} = \frac{45}{78},$$

Similarly, the probability that one white ball and one black ball are selected from the first bag containing 13 balls (in order to *transfer* to the second bag) is

$$P(A_2) = \binom{10}{1}\binom{3}{1} \div \binom{13}{2} = \frac{30}{78},$$

And, the probability that two black balls are selected from the first bag containing 13 balls (in order to *transfer* to the second bag) is

$$P(A_3) = \binom{3}{2} \div \binom{13}{2} = \frac{3}{78}.$$

AFTER having transferred 2 balls from the first bag, the second bag contains
i) 5 white and 5 black balls (if 2 white balls are transferred)

Colour of Ball	No. of Balls in Bag A	No. of Balls in Bag B
White	**10 – 2 = 8**	**3 + 2 = 5**
Black	**3**	**5**
Total	**13 – 2 = 11**	**8 + 2 = 10**

Hence: P(W/A1) = 5/10

ii) 4 white and 6 black balls (if 1 white and 1 black ball are transferred)

Colour of Ball	No. of Balls in Bag A	No. of Balls in Bag B
White	**10 – 1 = 7**	**3 + 1 = 4**
Black	**3 – 1 = 2**	**5 + 1 = 4**
Total	**13 – 2 = 11**	**8 + 2 = 10**

Hence: P(W/A2) = 4/10
iii) 3 white and 7 black balls (if 2 black balls are transferred)

Colour of Ball	No. of Balls in Bag A	No. of Balls in Bag B
White	**10**	**3**
Black	**3 – 2 = 1**	**5 + 2 = 7**
Total	**13 – 2 = 11**	**8 + 2 = 10**

Hence: P(W/A3) = 3/10

Let W represent the event that the WHITE ball is drawn from the second bag after having transferred 2 balls from the first bag.

Then $P(W) = P(A1 \cap W) + P(A2 \cap W) + P(A3 \cap W)$

Now $P(A1 \cap W) = P(A1)P(W/A1)$

$= 45/78 \times 5/10$

$= 15/52$

$P(A2 \cap W) = P(A2)P(W/A2)$

$= 30/78 \times 4/10$

$= 2/13,$

and

$P(A3 \cap W) = P(A3)P(W/A3)$

$= 3/78 \times 3/10$

$= 3/260.$

Hence the required probability is

$P(W)$

$= P(A1 \cap W) + P(A2 \cap W) + P(A3 \cap W)$

$= 15/52 + 2/13 + 3/260$

= 59/130
= 0.45

Next, we discuss the concept of INDEPENDENT EVENTS:

INDEPENDENT EVENTS:

Two events A and B in the same sample space S, are defined to be independent (or statistically independent) if the probability that one event occurs, is not affected by whether the other event has or has not occurred, that is

$P(A/B) = P(A)$ and $P(B/A) = P(B)$.

It then follows that two events A and B are independent if and only if

$P(A \cap B) = P(A)\ P(B)$

and this is known as the *special case* of the Multiplication Theorem of Probability.

RATIONALE:

According to the multiplication theorem of probability, we have:

$P(A \cap B) = P(A)\ .\ P(B/A)$

Putting $P(B/A) = P(B)$, we obtain

$P(A \cap B) = P(A)\ P(B)$

The events A and B are defined to be *DEPENDENT* if $P(A \cap B) \neq P(A) \times P(B)$.

This means that the occurrence of one of the events in some way affects the probability of the occurrence of the other event. Speaking of independent events, it is to be emphasized that two events that are independent, can NEVER be mutually exclusive.

EXAMPLE:

Two fair dice, one red and one green, are thrown.

Let A denote the event that the red die shows an even number and let B denote the event that the green die shows a 5 or a 6. Show that the events A and B are independent.

The sample space S is represented by the following 36 outcomes:

S = {(1, 1), (1, 2), (1, 3), (1, 5), (1, 6);
(2, 1), (2, 2), (2, 3), (2, 5), (2, 6);
(3, 1), (3, 2), (3, 3), (3, 5), (3, 6);
(4, 1), (4, 2), (4, 3), (4, 5), (4, 6);
(5, 1), (5, 2), (5, 3), (5, 5), (5, 6);
(6, 1), (6, 2), (6, 3), (6, 5), (6, 6) }

Since

A represents the event that red die shows an even number, and B represents the event that green die shows a 5 or a 6,

Therefore $A \cap B$ represents the event that red die shows an even number and green die shows a 5 or a 6.

Since A represents the event that red die shows an even number, hence P(A) = 3/6.

Similarly, since B represents the event that green die shows a 5 or a 6, hence P(B) = 2/6.

Now, in order to compute the probability of the joint event $A \cap B$, the first point to note is that, in all, there are 36 possible outcomes when we throw the two dice together, i.e.

S = {(1, 1), (1, 2), (1, 3), (1, 5), (1, 6);
(2, 1), (2, 2), (2, 3), (2, 5), (2, 6);
(3, 1), (3, 2), (3, 3), (3, 5), (3, 6);
(4, 1), (4, 2), (4, 3), (4, 5), (4, 6);
(5, 1), (5, 2), (5, 3), (5, 5), (5, 6);
(6, 1), (6, 2), (6, 3), (6, 5), (6, 6) }

The joint event $A \cap B$ contains only 6 outcomes out of the 36 possible outcomes.

These are (2, 5), (4, 5), (6, 5), (2, 6), (4, 6), and (6, 6).

and $P(A \cap B) = 6/36$.

Now

P(A) P(B)
= $3/6 \times 2/6$
= 6/36

$= P(A \cap B)$.

Therefore the events A and B are independent.

Let us now go back to the example pertaining to live births and stillbirths that we considered in the last lecture, and try to determine whether or not sex of the baby and nature of birth are independent.

EXAMPLE :

Table-1 below shows the numbers of births in England and Wales in 1956 classified by (a) sex and (b) whether live born or stillborn.

Table-1

Number of births in England and Wales in 1956 by sex and whether live- or still born.
(Source *Annual Statistical Review*)

	Liveborn	Stillborn	Total
Male	359,881 (A)	8,609 (B)	368,490
Female	340,454 (B)	7,796 (D)	348,250
Total	700,335	16,405	716,740

There are four possible events in this double classification:

- Male live birth,
- Male stillbirth,
- Female live birth, and
- Female stillbirth.

The corresponding relative frequencies are given in Table-2.

Table-2

Proportion of births in England and Wales in 1956 by sex and whether live- or stillborn.
(Source Annual Statistical Review)

	Liveborn	Stillborn	Total
Male	.5021	.0120	.5141
Female	.4750	.0109	.4859
Total	.9771	.0229	1.0000

As discussed in the last lecture, the total number of births is large enough for these relative frequencies to be treated for all practical purposes as *PROBABILITIES*.

The compound events ‘Male birth’ and ‘Stillbirth’ may be represented by the letters M and S.

If M represents a male birth and S a stillbirth, we find that

$$\frac{n(M \text{ and } S)}{n(M)} = \frac{8609}{368490} = 0.0234$$

This figure is the proportion — and, since the sample size is large, it can be regarded as the *probability* — of males who are still born — in other words, the *CONDITIONAL* probability of a stillbirth *given that* it is a male birth. In other words, the probability of stillbirths *in* males.
The corresponding proportion of stillbirths among females is

$$\frac{7796}{348258} = 0.0224.$$

These figures should be contrasted with the OVERALL, or *UNCONDITIONAL*, proportion of stillbirths, which is

$$\frac{16405}{716740} = 0.0229.$$

We observe that the conditional probability of stillbirths among boys is slightly *HIGHER* than the overall proportion. Where as the conditional proportion of stillbirths among girls is slightly *LOWER* than the overall proportion. It can be concluded that sex and stillbirth are statistically *DEPENDENT*, that is to say, the *SEX* of a baby yet to be born *has* an effect, (although a small effect), on its chance of being stillborn. The example that we just considered point out the concept of *MARGINAL PROBABILITY*.

Let us have another look at the data regarding the live births and stillbirths in England and Wales:
Table-2Proportion of births in England and Wales in 1956 by sex and whether live- or stillborn. (Source *Annual Statistical Review*)

	Liveborn	Stillborn	Total
Male	**.5021**	**.0120**	**.5141**
Female	**.4750**	**.0109**	**.4859**
Total	**.9771**	**.0229**	**1.0000**

And, the figures in Table-2 indicate that the probability of male birth is 0.5141, whereas the probability of female birth is 0.4859.Also, the probability of live birth is 0.9771, where as the probability of stillbirth is 0.0229.
And since these probabilities appear in the margins of the Table, they are known as *Marginal Probabilities*. According to the above table, the probability that a new born baby is a male and is live born is 0.5021 whereas the probability that a new born baby is a male and is stillborn is 0.0120.Also, as stated earlier, the probability that a new born baby is a male is 0.5141, and, CLEARLY, 0.5141 = 0.5021 + 0.0120.
Hence, it is clear that the joint probabilities occurring in any row of the table *ADD UP* to yield the corresponding *marginal* probability.
If we reflect upon this situation carefully, we will realize that this equation is totally in accordance with the Addition Theorem of Probability for mutually exclusive events.
P(male birth)
= P(male live-born *or* male stillborn)
= P(male live-born) + P(male stillborn)
= 0.5021 + 0.0120
= 0.5141

Another important point to be noted is that:
Conditional Probability
Joint Probability
Marginal Probability

EXAMPLE:

P(stillbirth/male birth)

P(male birth *and* stillbirth)/P(male birth)

=0.0120/0.5141

= 0.0233